*A Theory of Geometrical Relations.**

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INTRODUCTION.

In the present paper we aim primarily to establish the result that descriptive or projective order in n-dimensions ($n = 1, 2, 3, \ldots$) may be generated p-dimensionally ($p = 1, 2, 3, \ldots, n$), so that the corresponding geometric spaces $S_{p,n}$, $\Sigma_{p,n}$ may be said to have the index (p, n), where p denotes the dimensionality of the generating relation and n is the dimensionality of the space generated.† Each of the geometric spaces is generated by two indefinables, viz., the element point and the generating relation; the other indefinables, such as that of the ordered dyad, are more broadly logical. Provided $n \gtreqless 3$, each of the descriptive spaces $S_{p,n}$ is extensible to the usual n-dimensional projective geometry by well-known methods.‡

Among the p-dimensional relations which are effective for the generation of descriptive or projective n-space, those which involve a minimum number of points, viz., $p + 1$, are especially important. We designate such a relation, which is necessarily descriptive, by $\alpha_1 R_p \alpha_2 \ldots \alpha_{p+1}$, where $p = 1, 2, 3, \ldots$. This relational proposition may be interpreted concretely for $p = 1$ by, "α_1 precedes α_2," and hence R_1 is the well-known relation of Vailati;§ for $p = 2$ by, "if a person swims from α_2 to α_3, the point α_1 is at his right"; for $p = 3$ by, "to a person stationed at α_1 'motion' along the triangle $\alpha_2 \alpha_3 \alpha_4$ in the indicated

* Read before the American Mathematical Society under the following titles: "On the Foundations of Abstract Geometry," April, 1906; "Concerning Abstract Geometrical Relations," "Systems of Axioms for Projective Geometry," September, 1906; "On the Ausdehnungslehre of Grassmann," March, 1907; "On the Relation of Right-handedness in Geometry," September, 1907.

† The interesting memoir by F. Morley, "On the Geometry Whose Element is the 3-Point of a Plane," *Trans. Am. Math. Soc.* (1904), Vol. V, p. 467, has an important relation with our development.

‡ Klein, *Mathematische Annalen*, IV, VI; Pasch, "Vorlesungen über Neuere Geometrie"; Bonola, *Giornale di Matematiche*, XXXVIII.

§ Vailati, *Rivista di Matematica*, II, pp. 71–75.

order appears clockwise,"* hence R_3 is equivalent to a relation of (absolute) right-handedness. In general, the relation R_p is p-dimensionally transitive and alternating.†

The p-dimensional systems ${}^pR_p$ generated by the relation R_p are equivalent to the systems ${}^pK_p$ generated by the (p-dimensional) linearly transitive and symmetrical relation K_p between two ordered $(p+1)$-ads,‡ $\alpha_1 \alpha_2 \ldots . \alpha_{p+1} K_p \beta_1 \beta_2 \ldots . \beta_{p+1} (p = 1, 2, 3, \ldots .)$. For $p = 3$ the relation K_p expresses sameness of sense, or relative right-handedness. An important distinction between the systems ${}^pR_p$ and ${}^pK_p$ is seen when each system is extended to n-dimensions, $n > p$. For such an extension an infinite class of relations of the type R_p is required, whereas a single relation between $(p+1)$-ads, analogous to the relation K_p, suffices in case of the extension of the system ${}^pK_p$.

For $p \gtreqless 3$ the systems ${}^pK_p$ are made "complete," for euclidian geometry by adding a certain number of axioms which permit the introduction of the real number system in a very simple manner.§ These "complete" systems are shown to provide an elegant basis for Grassmann's Ausdehnungslehre and the exposition of the latter by Peano. ||

Finally we discuss, in addition to the relations R_p and K_p, their projective analogues and other p-dimensional relations which generate n-dimensional geometry; also we consider their mutual definition. For example, we show that the well-known relation of "betweenness" $\alpha_1 \frac{I_1}{B_1} \beta_1 \beta_2$ has the p-dimensional extensions $\alpha_1 I_p \beta_1 \beta_2 \ldots . \beta_{p+1}$ and $\alpha_1 \alpha_2 \ldots . \alpha_p B_p \beta_1 \beta_2$, where the order of the elements in the terms of each relation is immaterial. The former proposition expresses concretely, "α_1 is in the interior of the $p+1$ independent points $\beta_1 \beta_2 \ldots . \beta_{p+1}$"; the latter proposition expresses, "the p independent points $\alpha_1 \alpha_2 \ldots . \alpha_p$ are between β_1, β_2."¶

* Cf. Möbius, "Der barycentrische Calcul," *Gesammelte Werke*, I, §§ 17–20.

† For definitions, see Chapter III.

‡ If the $(p+1)$-ads are identical, the order of the elements is immaterial.

§ Cf. Grassmann, "Ausdehnungslehre," 1844, *Gesammelte Werke*, I, p. 138, note 1.

|| Peano, "Calcolo Geometrico;" *Formulaire Mathématique*, Turin (1903), pp. 277, 338; (1905), p. 188. Cf. also Schweitzer, "On the Logical Basis of Grassmann's Extensive Algebra," *Bulletin American Math. Soc.*, November, 1908.

¶ In connection with these relations we find it desirable to formulate descriptive axioms in terms of a relation $\alpha_1 \alpha_2 S \beta_1 \beta_2 \ldots . \beta_p$, which expresses that, "$\alpha_1 \alpha_2$ are on the same side of the independent points $\beta_1 \beta_2 \ldots . \beta_p$." A comparison of these "S" systems with our "R" systems is most instructive.

CHAPTER I.

Historical Remarks on the Theory of Relations.

Previous to De Morgan the subject of relations was considered almost wholly from a metaphysical standpoint. Thus De Morgan says: *

"Much has been written on relation in all its psychological aspects except the logical one, *i. e.*, the analysis of necessary laws of thought connected with the notion of relation. The logician has hitherto carefully excluded from his science the study of relation in general; he places it among those heterogeneous *categories* which turn the porch of his temple into a magazine of raw material mixed with refuse."

Again, on the inadequacy of the previous logical treatment of relations the same author says: †

"The only relations admitted into logic, down to the present time, are those which can be signified by *is* and denied by *is not*.... Accordingly, all logical relation is affirmed to be reducible to *identity*, A is A, to *non-contradiction*, Nothing both A and not-A, and to *excluded middle*, Everything either A or not-A. These three principles, it is affirmed, dictate all the forms of inference, and evolve all the canons of syllogism.... I cannot see how, alone, they are competent to the functions assigned."

Observing that one may ignore on the logical basis the metaphysical aspect of relations, De Morgan was the first to investigate their formal theory. ‡ His suggestive memoir on relations has been of marked influence on subsequent authors and contains features which have been generally recognized as fundamental. §

Although De Morgan first emphasized the formal theory of relations, it is due to Peirce || to have systematically developed his ideas and to have pointed out their adaptability to a calculus. The work of Peirce has been carried to a

* *Cambridge Philosophical Transactions*, 1864, p. 331.

On the definition of a relation J. S. Mill says (James Mill, "Analysis," II, 10): "Any objects, whether physical or mental, are related .... in virtue of any complex state of consciousness into which they both enter, even if it be a no more complex state of consciousness than that of thinking them together." Compare De Morgan, *l. c.*, p. 208; Peirce, *American Journal*, III, p. 42.

† *l. c.*, p. 335.

‡ "On the Syllogism I, II, III, IV," *Camb. Phil. Trans.*, 1847-64.

§ Cf. Peirce, *American Journal*, VII, pp. 201, 202; III, p. 21, etc.

|| *American Journal*, III.

high degree of elaboration by Schroeder.* On the plan of his discussion of relations the latter says:†

"Bei der fast unermesslichen Mannigfaltigkeit der Richtungen, nach welchen sich die Disziplin entwickelungsfähig zeigt, der Fülle ihrer Anwendungsmöglichkeiten auf die verschiedensten Gebiete—zu denen die Begriffe von 'Endlichkeit', 'Anzahl', 'Funktion' und 'Substitution' ebensowohl gehören als wie z. B. die 'menschlichen Verwandtschaftsverhältnisse' —, bei ihrer Doppelnatur als einer *Algebra* einerseits und einer Entwickelungsform der *Logik* andrerseits, nämlich ihrer *Ausgestaltung zur Logik der Beziehungen* (*und Beziehungsbegriffe*, 'Relative') *überhaupt*, scheint es unerlässlich — soll nicht die Uebersicht leiden und der Eindruck der Schönheit und Konsequenz des Ganzen verloren gehen — dass wir die verschiedenen Gesichtspunkte, unter welchen unsere Theorie zu betrachten sein wird, thunlichst scharf von einander getrennt halten. Ich werde deshalb zunächst *eine* Seite der Theorie fast ausschliesslich bevorzugen, und zwar dieselbe lediglich als eine *Algebra*, einen *Kalkul* aufbauen.... Erst wenn auf diesem Wege ein gewisser Grundstock geschaffen und ein schon recht ansehnliches Kapital von absolut feststehenden Wahrheiten — Thatsachen der Deduktion — gesichert ist, gedenke ich .... auf die Fundamente der Disziplin zurück zu kommen, um deren zuerst nur einfach hingestellte Festsetzungen dann auch heuristisch zu motiviren und aus allgemein logischen Gesichtspunkten reflektirend zu erörtern, insbesondere sie als den Zwecken eben dieser Wissenschaft, der Logik, dienstbare nachzuweisen.... Meine Bezeichnungsweisen schliessen sich sehr nahe an die von Peirce‡ in *einer* seiner Abhandlungen gebrauchten an und werden die Abweichungen späterhin gekennzeichnet und gerechtfertigt."

The genesis of a "binary Relative" is thus given by Schroeder:§

"Die Disziplin geht aus von der Betrachtung eines Denkbereiches 1^1, bestehend aus '*Elementen*' A, B, C, ..., die als einander gegenseitig ausschliessend und von dem Nichts (0) verschieden vorausgesetzt werden. Als Inbegriff dieser Elemente wird der Bereich mittelst

$$\begin{aligned} 1^1 &= A + B + C + \dots \\ &= \Sigma_i \, i \end{aligned}$$

* "Vorlesungen über die Algebra der Logik."

† *l. c.*, III, 1 (1895), p. 1.

‡ Schroeder, *l. c.*, I, p. 710, Note 9c.

§ *Mathematische Annalen*, XLVI, p. 144.

in Gestalt von deren 'indentischer Summe' (logical aggregate) dargestellt. Doch ist sogleich zu betonen, dass diese Elemente — wie die (reellen) Zahlen oder die Punkte einer Geraden (ev. Strecke) — auch ein Kontinuum bilden dürfen. Irgend zwei Elemente i und j lassen sich — etwa unter dem Gesichtspunkt einer gewissen von i zu j bestehenden '*Beziehung*' — in bestimmter Folge zu einem *Elementepaar* (oder individuellen binären Relative) $i:j$ zusammenstellen, und bildet die Gesammtheit aller erdenklichen Elementepaare

$$1^2 = \Sigma_{ij}\, i:j$$

einen zweiten aus dem ursprünglichen abgeleiteten Denkbereich, der aus den Variationen mit Wiederholungen zur zweiten Klasse von des letzteren Elementen besteht. In diesem zweiten Bereich bewegt sich unsere ganze Disziplin, und es wird unter einer binären Relative (a oder b, c, ...) nichts anderes zu verstehen sein, als ein Inbegriff (identische Summe) von Elementenpaaren (keinen, einigen, oder allen) irgendwie hervorgehoben aus genanntem Bereiche."

That is, a class of elements gives rise to a class of ordered dyads,* and one obtains a relation, say $\alpha Q \beta$, satisfied by every ordered dyad $\alpha\beta$ of a class of dyads derived from the initial class in virtue of given *principles of selection.*†

Schroeder's treatment of relations contains many valuable contributions, such as the preceding generation of a binary relative; but, as he himself indicates, it is primarily an independent discipline, an algebra of relations. It is due to Russell‡ to have developed a theory of relations which is at once free from a complicated symbolism and, as he shows, broadly accessible to mathematics. In accomplishing this task Russell has been very materially aided by the investigations of Peano.§

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* On the meaning of an ordered dyad, see Chapter II.

† Royce, *Transactions of the American Mathematical Society*, VI, p. 353, has also generated relations on the selective basis.. Cf. also Schroeder, "Algebra der Logik," Vol. II, 2 (1905), Anhang 8. It is an interesting problem to arrive at our "R" or "K" systems by means of purely selective methods on the basis of a suitable existential domain; undefined principles of selection will then take the place of our formal indefinables.

‡ See his "Principles of Mathematics."

§ Cf. A. T. Shearman, "The Development of Symbolic Logic," Chapter VI. The general logical position of our paper is formal (cf. De Morgan, Russell, *l. c.*), not selective. The reader will find it interesting to compare our paper with the following sections of "Russell": §§ 53–55, 71, 81–82 (cf. 54), 83 (last lines of p. 87), 89, 96, 98, 187–208, 222, 225, etc.

CHAPTER II.

*On the Theory of n-Dimensional Chains** ($n = 1, 2, 3, \ldots.$).

In the following, as indeed throughout the present paper, we assume as logical indefinable the functional ordered dyad xy in the variables x, y. A functional ordered dyad xy has the property that xy is a proper ordered dyad when x and y are given specific values from their respective ranges. If $a \neq b$, then the ordered dyads ab, ba are distinct; also if $c \neq a$ or b, or $d \neq a$ or b, the ordered dyads ab and cd are distinct. Thus the necessary and sufficient condition that $ab = cd$ is $a = c$ and $b = d$. Also the elements of an ordered dyad are not necessarily distinct.

Of the dyads ab, ba any one is called the conjugate of the other. The dyad aa is conjugate to itself. The dyads ax, xb are said to be connected, ab is their resultant and x is their element of connection. We define now a linear permutation of the elements $a_1, a_2, \ldots, a_n$ to be the set of ordered dyads formed out of these elements with the following properties:

1. Every element appears in dyads as the associate of the remaining elements.
2. A dyad and its conjugate are not both in the set.
3. If any two connected dyads are in the set, then their resultant is also in the set.

A linear permutational set in n elements $a_1, a_2, \ldots, a_n$ is then of the following type:

$$\begin{array}{llll}
a_1 a_2 & a_2 a_3 & a_3 a_4 \ldots . a_{n-1} a_n \\
 & a_1 a_3 & a_2 a_4 \ldots . a_{n-2} a_n \\
 & & a_1 a_4 \ldots . a_{n-3} a_n \\
 & & \ldots\ldots\ldots \quad \ldots . \\
 & & \qquad\qquad a_1 a_n
\end{array}$$

Two permutations are identical if they contain the same elements and the dyads of one set can be identified with the dyads of the other set.

A set of elements S gives rise to a set of dyads called a linear chain under the following conditions:

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* On the term "chain," as used in this chapter, compare article on Kinematics in *Encyclopaedie der Mathematischen Wissenschaften*, IV, 3, § 28.

1. Every element appears in at least one dyad of the set.
2. A dyad and its conjugate are not both in the set.
3. Every dyad is connected with some other dyad.
4. If there exists a dyad connected in only one element, there exists at most one other such dyad.
5. No element appears in more than two dyads of the set.

These conditions are satisfied by the following sets of dyads:

1) *ab* *bc* *ca* *ef* *fg*,
2) *ab* *bc* *ca* *ef* *fg* *ge*.

The conditions are also necessary, as is shown by these systems:

1. This condition is obviously independent.
2. *ab* *bc* *cd* *ef* *fe*.
3. *ab* *bc* *ca* *ef*.
4. *ab* *bc* *cd* *ef* *fg*.
5. *ab* *bc* *ca* *ce*.

In order that the chain of dyads satisfying 1–5 be unique it is necessary to add:*

6. If a set of dyads satisfies 1–5 it also satisfies the condition: The set of elements S is contained by any set T of elements with the following properties:

1) If there is a dyad ab such that a is not an element of connection, then T contains an element a; if there is no such dyad, then T contains some element of S.

2) T contains every element x of S such that some element u common to S and T forms with x a dyad ux in the set of dyads.

Under postulates 1–6 two cases may arise: there exists a dyad which is connected in only one element, or there is no such dyad. In the former case the chain is said to be open; in the latter, it is closed. An open chain in a finite number of elements contains two dyads, ab, kl, such that a and l are not elements of connection; a and l are called the first and last elements of the chain respectively. There is, moreover, a unique second element in an open chain, a unique third, etc. An important property is the following: Every linear permutation contains uniquely an open linear chain in the same number of elements. Conversely, given an open linear chain of dyads in a finite number of elements we may arrive at a permutational set of dyads by adding to the set of dyads of the

* Cf. Russell, "The Principles of Mathematics," §§ 189, 229.

chain the resultants of the connected dyads in the chain, adding the resultants of the connected dyads contained in the set thus obtained, and so on.

In an interesting memoir on tactical systems E. H. Moore* has given a valuable generalization of a closed linear chain by means of his systems $S\{k, l, m\}$. In fact, every closed linear chain is a system $S\{2, 1, m\}$, but not necessarily conversely; since, for example, the dyads 12, 21, 34, 43 are an $S\{2, 1, 4\}$, and the dyads 12, 23, 31, 45, 56, 64 are an $S\{2, 1, 6\}$, but contradict postulates 2 and 6.

An important generalization of open and closed linear chains may be made in the following manner. By a planar triad $[abc]$ we mean the dyads cb, ac, ba. The triad $[abc]$ has a unique conjugate, namely, $[bac]$. The three-dimensional tetrad $[abcd]$ is the set of planar triads $[dbc]$, $[adc]$, $[abd]$, $[bac]$; and $[bacd]$ is the conjugate of $[abcd]$. In general, we define the $(p-1)$-dimensional p-ad $[a_1 a_2 \ldots . a_p]$ to be the set of $(p-2)$-dimensional $(p-1)$-ads $[a_p a_2 \ldots . a_{p-1}]$ $[a_1 a_p \ldots . a_{p-1}] \ldots . [a_1 a_2 \ldots . a_{p-2} a_p]$ $[a_2 a_1 \ldots . a_{p-1}]$; and $[a_2 a_1 \ldots . a_{p-1} a_p]$ is the conjugate of $[a_1 a_2 \ldots . a_p]$. The $(p-1)$-dimensional p-ads $[x a_2 \ldots . a_p]$ $[a_1 x \ldots . a_p] \ldots . [a_1 a_2 \ldots . x]$ are connected, x is the element of connection and $[a_1 a_2 \ldots . a_p]$ is their resultant. A set of connected $(p-1)$-dimensional p-ads $(p = 2, 3, \ldots .)$ may be exhibited in the following formal way:

$$\begin{array}{ccccccccc} & & & [ax] & & [xb] & & & \\ & & [axb] & & [bxc] & & [cxa] & & \\ & [acxb] & & [bxdc] & & [cdax] & & [xabd] & \\ [bcxde] & & [edxba] & & [abxec] & & [cexad] & & [daxcb] \\ & & & & \text{etc., etc.} & & & & \end{array}$$

If p is odd each set is obtained from a definite set of closed linear chains and the element x by a simple process of formal interpolation; if p is even, each set is obtained from a definite set of open linear chains by a similar interpolation.† To construct now p-dimensional open chains we proceed thus. We illustrate the construction for $p = 2$, although the method is easily recognized to be general. With every planar triad $[abc]$ three systems of connected triads can be constructed in the elements x, a, b, c, each system containing the triad $[abc]$. For we need only find the sets of connected triads of which each of the triads

* *American Journal*, Vol. XVIII, p. 268.

† The p-dimensional open and closed chains which we shall construct below are made accessible to the substitution theory by exhibiting them formally in a manner analogous to the above exhibition of connected p-ads.

$[xbc]$, $[axc]$, $[abx]$ is the resultant respectively; we get then

$$\begin{matrix} [abc] & [xac] & [xba] \\ [bxc] & [abc] & [axb] \\ [cxb] & [acx] & [abc] \end{matrix}$$

Consider now the triad $[abc]$ and let $x \neq a, b, c$. This triad gives rise to an open chain of triads if 1) $[abc]$ is replaced by the three triads $[xbc]$, $[axc]$, $[abx]$, of which $[abc]$ is the resultant, 2) or to the triad $[abc]$ is added any one of the three sets of two triads which form with $[abc]$ a set of connected triads. Thus an open planar chain in four elements a, b, c, x consists of three connected triads and the triad $[abc]$ gives rise to four distinct chains in four elements. Likewise the conjugate triad $[bac]$ will give rise to four distinct chains, each of which is distinct from the former, so that the total number of planar chains in four elements is $2 . 4 = 8$. Each of the preceding chains may be lengthened in a manner analogous to the above, viz., we replace any triad of a chain by the three triads of which it is the resultant or we add to the given chain of triads any one of the three sets of two triads which are connected with the boundary of the chain of triads. The boundary of a chain of triads is that triad which gives rise to the given chain under the following successive processes:

1. Replacing the triad by a set of triads of which it is the resultant;
2. Replacing some one of the set of triads thus obtained by a corresponding set of connected triads; and so on.

The boundary of a set of connected triads is evidently their resultant.

It is easily seen that five elements give rise to $2 . 4 . 6 = 48$ planar chains and that the total number of planar chains in n elements ($n = 3, 4, \ldots.$) is

$$2 . 4 . 6 \ldots . 2 (n-2) = 2^{n-2} . (n-2)!$$

For three-dimensional chains the discussion analogous to that for planar chains may now be carried out. We find that the complete number of spatial chains in n-elements ($n = 4, 5, 6, \ldots.$) is $2 . 5 . 8 \ldots . (3n-10)$. The preceding is also easily extended to p-dimensions $p \geqq 1$. We may show that the complete number of p-dimensional chains in n-elements $1 \leqq p < n$ is

$$N = (0 . p + 2)(1 . p + 2)(2p + 2) \ldots . ([n - p - 1] p + 2).$$

Thus, for $p = 1$, $N = n!$; $p = 2$, $N = 2^{n-2}(n-2)!$; etc.

The preceding p-dimensional chains were open; if to each such chain we add the conjugate of its boundary, we get closed chains.

From a p-dimensional chain arises a p-dimensional permutation by adding to the $(p+1)$-ads of the chain the resultants of the connected $(p+1)$-ads in the chain, adding the resultants of the connected $(p+1)$-ads contained in the set thus obtained, and so on. Of course, a p-dimensional permutation may be formally defined independently of a p-dimensional chain, and vice versa. Evidently a p-dimensional permutation in n elements contains a p-dimensional chain in the same number of elements. The formal theory of p-dimensional permutations is, moreover, intimately related* with the p-dimensional generation of p-space, which is discussed in Chapter IV.

CHAPTER III.

On the General Theory of Relations, Classes, and Operations.

As formal types of relational, class, and operational propositions we take aRb, $a\varepsilon C(b)$, $O_b(a)$ respectively. These propositions may be considered as expressing, "a possesses the relation R with reference to b," "a is in the class C with reference to b," "the operation O with reference to b affects a." With respect to the interdependence of the preceding propositions, we assume that

$$aRb \sim a\varepsilon C(b) \sim O_b(a);$$

that is, they are equivalent to one another.†

We assume the following analysis of the above propositions:

$$aRb = aR + b = a + Rb,$$
$$a\varepsilon C(b) = aC + b = a + Cb,$$
$$O_b(a) = Oa + b = a + bO.$$

The symbol of composition "$+$" is an indefinable which implies no ordering of the terms. The symbols aR, aC, Oa may be interpreted concretely by, "a possesses the relation R," "a is in the class C," "the operation O affects a." The symbols Rb, Cb, bO may be interpreted by, "the relation R refers to b," "the class C refers to b," "the operation O refers to b." We call aR and Ra the regressive and progressive relational associates of a respectively; etc. Two such associates are always distinct; and two regressive (progressive) associates

* For example, consider the planar permutation in 5 elements:

[423] [543] [153] [145] [124] [143] [123].

In the plane we have then the transitive property that [423] [143] [124] and [543] [153] [145] imply [523] [153] [125].

† Professor George H. Mead suggests that it may be better to assume *aRb implies* $a\varepsilon C(b)$ *implies* $O_b(a)$.

bR, cR (Rb, Rc) are identical only if their terms b, c are identical. Further, if aRb, then for any relational associate Rc, a is distinct from Rc. Thus if aRb and cRd are identical, we must have $a = c$, $Rb = Rd$; similarly, $b = d$, $aR = cR$; that is, $a = c$ and $b = d$. Hence if $a \neq b$, aRb and bRa are distinct. The terms of a relational proposition have, therefore, the character of an ordered dyad.*

We consider the equivalence

$$aRb \sim a\varepsilon C(b) \sim O_b(a).$$

Let $a'b'$ and ab be distinct dyads and suppose that for one of the members of the above equivalence, say aRb, we have $a'Rb'$; then we assume that

$$a'Rb' \sim a'\varepsilon C(b') \sim O_{b'}(a').$$

Let us assume also that

$$aRb \sim abR_1 ab.$$

Then ab is an ordered dyad. Finally, if $aRb \sim abR_1 ab$, and $a'b' \neq ab$, we assume that

$$a'Rb' \sim a'b'R_1 a'b'.$$

Thus from the preceding it follows that for any two terms x, y such that xRy, the latter proposition is equivalent to $xyR_1 xy$ and therefore to $xy\varepsilon C(xy)$. That is, every relation can be interpreted in terms of a class of ordered dyads.

By the ordered n-ad $a_1 a_2 \ldots a_n$, $n \overline{\gtrless} 2$ we mean some one of the sets of ordered dyads obtained from the open linear chain $x_1 x_2 \; x_2 x_3 \ldots x_{n-1} x_n$ of dyads in n distinct variables. Thus the elements of an ordered n-ad are not necessarily distinct.

Now the terms of the proposition aRb may be, in particular, an n-ad and a p-ad. If the n-ad and the p-ad are identical the relational proposition is of the type $a_1 a_2 \ldots a_n R a_1 a_2 \ldots a_n$, which we write $a_1 a_2 \ldots a_n R_{12\ldots n}$, the subscript of the R being a linear open chain. In this case we say that the $a_1, a_2, \ldots, a_n$ are mutually related. We postulate that from the preceding proposition $n!$ derived (and equivalent) propositions exist, viz.,

$$\{ a_{j_1} a_{j_2} \ldots a_{j_n} R_{j_1 j_2 \ldots j_n} \},$$

including the identity, *i. e.*, the given proposition. The relations

$$\{ R_{j_1 j_2 \ldots j_n} \}$$

are called the derived relations of the relation $R_{12\ldots n}$. In particular, if $n = 2$, R_{21} is the conjugate of the relation R_{12}. $R_{j_1 j_2 \ldots j_n}$ is an even derived relation

* We need hardly say that the analysis in this paragraph and the equivalence in the preceding paragraph are speculative. The reader may compare our remarks with Russell, *l. c.*, §§ 38, 71, 76, 77, 96, 212–215.

of $R_{12\ldots.n}$ if the $(n-1)$-dimensional n-ads (cf. Chapter II) $[12\ldots.n]$ and $[j_1 j_2 \ldots . j_n]$ are identical; otherwise $R_{j_1 j_2 \ldots . j_n}$ is an odd derived relation of $R_{12\ldots.n}$. If all the derived relations of a given relation are identical, then the given relation is symmetrical; if only the even derived relations are identical, the given relation is alternating.*

Mutual relations may or may not be transitive. Transitiveness is either linear, or planar, ... or n-dimensional. We define the mutual relation R to be n-dimensionally transitive ($n = 1, 2, 3, \ldots.$) if the propositions (dropping for convenience the subscript of the R)

$$\begin{gathered} x\, a_2 \ldots . a_{n+1} R, \\ a_1 x \ldots . a_{n+1} R, \\ \ldots\ldots\ldots\ldots, \\ a_1 a_2 \ldots\ldots x R \end{gathered}$$

imply

$$a_1 a_2 \ldots . a_{n+1} R.$$

The analogous definition in the case of relations which are not mutual is easily found.

If $a_1 a_2 \ldots . a_n R_{12\ldots.n}$ is a mutual relation, we assume that it is equivalent to the mutual relation

$$b_1 b_2 \ldots . b_k\, Q_{12\ldots.k}^{j'_1 j'_2 \ldots . j'_k},$$

where

$$\begin{aligned} b_1 &= a_1 a_2 \ldots . a_{j_1} & j'_1 &= j_1, \\ b_2 &= a_{j_1+1} a_{j_1+2} \ldots . a_{j_2} & j'_2 &= j_2 - j_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ b_k &= a_{j_{k-1}+1} a_{j_{k-1}+2} \ldots . a_{j_k} & j'_k &= j_k - j_{k-1}, \qquad (j_k = n) \end{aligned}$$

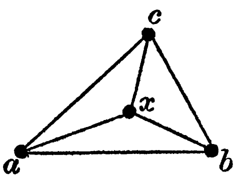
the numbers $j_1 j_2 \ldots . j_{k-1}$ being chosen arbitrarily. It is also assumed that the mutual relation $a_1 a_2 \ldots . a_n R_{12\ldots.n}$ is equivalent to each of the following relations:

$$\begin{gathered} a_1 R^{(1)} a_2 \ldots . a_n, \\ a_1 a_2 R^{(2)} a_3 \ldots . a_n, \\ \ldots\ldots\ldots\ldots\ldots, \\ a_1 a_2 \ldots . a_{n-1} R^{(n-1)} a_n. \end{gathered}$$

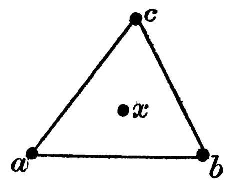
An illustration of the latter equivalence is easily given. Consider a point x in the interior of the triangle abc. We may look upon these four points as

* For the suggestion to use this term in the present connection we are indebted to Professor E. H. Moore. If the relation $R_{123}\ldots._n$ is alternating, then the alternating relation $R_{213}\ldots._n$ is called the conjugate of $R_{12}\ldots._n$, and conversely. Such a conjugate will sometimes be denoted by $\breve{R}$.

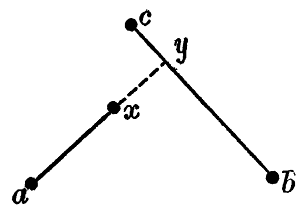
constituting a system; the points are then mutually related, *i. e.*, we have $xabc\,I\,xabc$. Then the statement that x is in the interior of abc expresses



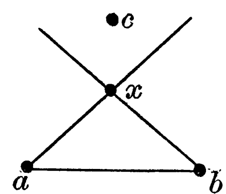
$xabc\,I\,xabc$



$x\,I^{(1)}\,abc$



$xa\,I^{(2)}\,bc$



$xab\,I^{(3)}\,c$

$x I^{(1)} abc$. $xa I^{(2)} bc$ is implied by the existence of y such that y is in the interior of bc and x is in the interior of ay. Finally, $xab\,I^{(3)} c$ is expressed by the statement that the compartment of the triangle xab at the vertex x has c in its interior.

CHAPTER IV.*

Geometrical Relations: The Systems ${}^nR_n$ $(n = 1, 2, 3, \ldots.)$.

In the present chapter we give systems of axioms for descriptive geometry in terms of the relation $R_n\,(n = 1, 2, 3, \ldots.)$. For $n \gtreqless 3$ each of the corresponding systems is sufficient for projective geometry if an axiom of continuity is added; the discussion of the systems thus amplified is, however, reserved for a later chapter. Under the respective axioms, the relation R_n involves $n+1$ independent points and therefore it may be termed n-dimensional. Since the generating relation R_n is n-dimensional and involves precisely $n+1$ points, we call it fundamental. The relation R_n is, moreover, n-dimensionally transitive and alternating (cf. Chapter III).

* In the original exposition of the system ${}^3R_3$ the author used the expression $\alpha\beta\gamma\delta = \Omega'$ to denote $\alpha R\beta\gamma\delta$. The expression $\alpha\beta\gamma\delta = \Omega'$ may be read "the tetrad $\alpha\beta\gamma\delta$ is in the class Ω'." Further, $\alpha\beta\gamma\delta = \Omega''$ was defined to be $\beta\alpha\gamma\delta = \Omega'$, and by $\alpha\beta\gamma\delta = \Omega$ was denoted, $\alpha\beta\gamma\delta = \Omega'$ or Ω''. If $\alpha\beta\gamma\delta = \Omega'$ and $\alpha'\beta'\gamma'\delta' = \Omega'$, then $\alpha\beta\gamma\delta = \alpha'\beta'\gamma'\delta'$, which may be read "$\alpha\beta\gamma\delta$ and $\alpha'\beta'\gamma'\delta'$ are in the same class," or, "the points $\alpha, \beta, \gamma, \delta$ and $\alpha', \beta', \gamma', \delta'$ are in the same order"; similarly, if $\alpha\beta\gamma\delta = \Omega''$ and $\alpha'\beta'\gamma'\delta' = \Omega''$, then $\alpha\beta\gamma\delta = \alpha'\beta'\gamma'\delta'$. If $\alpha\beta\gamma\delta = \Omega'$ and $\alpha'\beta'\gamma'\delta' = \Omega''$, then $\alpha\beta\gamma\delta \neq \alpha'\beta'\gamma'\delta'$; and similarly if $\alpha\beta\gamma\delta = \Omega''$ and $\alpha'\beta'\gamma'\delta' = \Omega'$, $\alpha\beta\gamma\delta \neq \alpha'\beta'\gamma'\delta'$; *i. e.*, "the tetrads $\alpha\beta\gamma\delta$ and $\alpha'\beta'\gamma'\delta'$ are in opposite classes" or "the points $\alpha, \beta, \gamma, \delta$ and $\alpha', \beta', \gamma', \delta'$ are in different orders." The expression $\alpha\beta\gamma\delta = \Omega'$ may be read "the points $\alpha, \beta, \gamma, \delta$ are in the order $\alpha\beta\gamma\delta$." With reference to the term "order," as here used, compare O. Veblen, *Transactions Am. Math. Soc.*, July, 1904. For the suggestion to employ, instead of the symbols "=", "≠", *uniformly* a relation K, the author is indebted to Professor E. H. Moore. The author takes pleasure, in this connection, in acknowledging the stimulation of Professor Moore's suggestion in the preparation of the present paper.

The systems ${}^nR_n$ will retain validity if for the generating relation $\alpha_1 R_n \alpha_2 \ldots \alpha_{n+1}$ is substituted the well-known outer (alternating) product of $n+1$ points $[\alpha_1 . \alpha_2 \ldots \alpha_{n+1}]$ due to Grassmann.* In this way the systems become a fundamental part of Grassmann's calculus.

System ${}^1R_1$. I. *Axioms.*

1. There exists α.†

2. The existence of α implies the existence of α_0, β_0 such that $\alpha_0 R \beta_0$ or $\beta_0 R \alpha_0$.

3. $\alpha R \beta$ implies $\beta \bar{R} \alpha$.‡

4. $\alpha R \beta$ and $\xi \neq \alpha, \beta$ imply $\alpha R \xi$ or $\xi R \beta$.

5. $\alpha R \beta$ and $\alpha R \xi$ and $\xi \neq \beta$ imply $\xi R \beta$ or $\beta R \xi$.

6. $\alpha R \beta$ implies the existence of ξ_1 such that $\beta R \xi_1$.

7. $\alpha R \beta$ implies the existence of ξ_2 such that $\xi_2 R \alpha$.

8. $\alpha R \beta$ implies the existence of ξ such that $\alpha R \xi$ and $\xi R \beta$.

II. *Definitions.*

1. ξ is on $\alpha\beta$ means, $\alpha R \beta$ and ($\alpha R \xi$ or $\xi R \beta$).

2. ξ is in the interior of $\alpha\beta$ means, $\alpha R \beta$ and $\alpha R \xi$ and $\xi R \beta$.

That is, we obtain one definition from the other by the proper interchange of "or" and "and".

III. *Theorems.*

1. $\alpha \neq \beta$ implies $\alpha R \beta$ or $\beta R \alpha$.

By axioms 1 and 2, $\alpha_0 R \beta_0$ or $\beta_0 R \alpha_0$. Suppose $\alpha_0 R \beta_0$. Then if the points α, β are the points α_0, β_0, the theorem is true. We distinguish, therefore, these cases:

1) $\alpha \neq \alpha_0, \beta_0$; $\beta \neq \alpha_0, \beta_0$. Since $\alpha_0 R \beta_0$ and $\alpha \neq \alpha_0, \beta_0$, by axiom 4, we have $\alpha R \beta_0$ or $\alpha_0 R \alpha$. If $\alpha R \beta_0$, since $\beta \neq \alpha, \beta_0$, then $\beta R \beta_0$ or $\alpha R \beta$. If $\alpha R \beta_0$ and $\beta R \beta_0$, then $\alpha R \beta$ or $\beta R \alpha$, by theorem 2. If $\alpha_0 R \alpha$, since $\beta \neq \alpha_0, \alpha$, we have $\beta R \alpha$ or $\alpha_0 R \beta$. If $\alpha_0 R \beta$, then by axiom 5, $\alpha_0 R \alpha$ and $\alpha_0 R \beta$ imply $\alpha R \beta$ or $\beta R \alpha$.

2) $\alpha = \alpha_0$ or β_0, $\beta \neq \alpha_0, \beta_0$. That is, we have $\alpha R \beta_0$ or $\alpha_0 R \alpha$. Proof as under 1).

* Cf. *Gesammelte Werke*, "Ausdehnungslehre," 1844 and 1862. For further details of the development from this view-point, see Chapter VI.

† Greek letters are used to denote points unless otherwise specified.

‡ The rule over the R is a symbol of negation.

3) $\beta = \alpha_0$ or β_0, $\alpha \neq \alpha_0, \beta_0$. That is, $\beta R \beta_0$ or $\alpha_0 R \beta$. Proof analogous to that under 1).

2. $\alpha R \beta$ and $\xi R \beta$ and $\xi \neq \alpha$ imply $\alpha R \xi$ or $\xi R \alpha$. By axiom 7 there is a point γ such that $\gamma R \xi$. Then assuming $\alpha \neq \gamma$, we get by axiom 4, $\gamma R \alpha$ or $\alpha R \xi$. If $\gamma R \alpha$, then since $\gamma R \xi$ we have by axiom 5, $\xi R \alpha$ or $\alpha R \xi$.

3. $\alpha R \xi$ and $\xi R \beta$ imply $\alpha R \beta$. By axiom 3, $\xi \neq \alpha, \beta$, $\beta \neq \alpha$. Then since $\alpha R \xi$ and $\beta \neq \alpha, \xi$, by axiom 4, $\beta R \xi$ or $\alpha R \beta$. Since $\xi R \beta$, by axiom 3 $\beta R \xi$ is impossible and $\alpha R \beta$ is true.

We have thus shown that our linear system ${}^1R_1$ implies the system due to Vailati.* Our system has, however, advantages which arise from the particular form in which it is stated. Namely, let us inquire whether the system ${}^1R_1$ is extensible to higher dimensions or not. That is, if we omit axiom 4 from the system ${}^1R_1$ and leave the remaining axioms in force, is it possible to add axioms in terms of the relation R such that a geometry of dimensionality greater than unity results, and what is the character of this geometry? In answering this question, we observe first that there exists a finite system of elements which satisfies all the preceding axioms except axiom 4; indeed, it is the following system† of 21 dyads in 7 elements:

12	23	34	45	56	67	71
25	57	74	41	13	36	62
51	16	64	42	27	73	35

Thus the dyads form three linear closed chains; the dyads of any two closed chains can be obtained from the third by means of the substitution

$$(4) \qquad (125) \qquad (376).$$

Every element is common to six dyads, and on every dyad lie five elements. The dyads may be arranged into a system of 14 (planar) triads:

[712]	[625]	[351]
[234]	[574]	[164]
[456]	[413]	[427]
[671]	[362]	[735]
[125]	[367]	

* Vailati, *Rivista di Matematica*, Vol. II, pp. 71–75.

† The extension of this system to n dimensions ($n > 1$) we shall discuss elsewhere. The above system is one of an infinitude of analogous systems. Cf. *Bulletin Am. Math. Soc.*, March, 1908, p. 265.

and into the following system of 21 (linear) triads:

$\overline{162}$	$\overline{273}$	$\overline{364}$	$\overline{425}$	$\overline{516}$	$\overline{627}$	$\overline{741}$
$\overline{235}$	$\overline{567}$	$\overline{734}$	$\overline{451}$	$\overline{123}$	$\overline{356}$	$\overline{642}$
$\overline{571}$	$\overline{136}$	$\overline{674}$	$\overline{412}$	$\overline{257}$	$\overline{713}$	$\overline{345}$

It will be observed that each dyad occupies all possible positions in the triads once and only once, thus occuring in three triads. Hence the preceding system is a subset of Moore's 3-idic system $S\{3, 2, 7\}$.* It is easily verified that under the preceding system of dyads all the axioms of the system ${}^1R_1$ are effective, *i. e.*, their hypotheses are fulfilled; and that axiom 4 is contradicted, the remaining axioms being satisfied. The geometry which is represented by the preceding system of dyads is not, however, a descriptive system; consider, for example, the triangle 125: we have 3 on 12 and 25. Let us require, then, that the geometry represented by the extended system of axioms be descriptive. We define, as above:

ξ is on $\alpha\beta$ means, $\alpha R \beta$ and ($\alpha R \xi$ or $\xi R \beta$),† and hence ξ not on $\alpha\beta$ means, $\alpha R \beta$ and $\alpha \bar{R} \xi$, $\xi \bar{R} \beta$.

Let us see in what particular form we must put this definition. Suppose that ξ is not on $\alpha\beta$ and that

$$\alpha R \beta, \qquad \alpha R_1 \xi, \qquad \xi R_2 \beta.$$

Then since our geometry is to be descriptive and we have ξ not on $\alpha\beta$, we must have, also, β not on $\alpha\xi$, α not on $\xi\beta$; that is,

$$\beta \bar{R}_1 \xi, \qquad \alpha \bar{R}_1 \beta,$$
$$\alpha \bar{R}_2 \beta, \qquad \xi \bar{R}_2 \alpha.$$

Different cases evidently arise according as R, R_1, R_2, $\breve{R}$, $\breve{R}_1$, $\breve{R}_2$ are distinct or not; these are in essence as follows:

1) R_1 or $R_2 = \breve{R}$; *i.e.*, $\xi R \alpha$ or $\beta R \xi$. If $\alpha R \beta$ and $\xi R \alpha$, then let $\eta' R \alpha$ and $\eta' R \beta$; hence η' is on $\xi\alpha$ and $\alpha\beta$, which is impossible since $\eta' \neq \alpha$. If $\alpha R \beta$ and $\beta R \xi$, let $\beta R \eta''$ and $\alpha R \eta''$. Then η'' is on $\alpha\beta$ and $\beta\xi$ and $\eta'' \neq \beta$. This is impossible.

* Cf. E. H. Moore, "Tactical Memoranda," *American Journal*, Vol. XVIII, pp. 268, 270. The preceding system of fourteen planar triads may be looked upon as two *triple systems* in seven elements. This connection was kindly pointed out to us by Professor Moore.

† *I. e.*, for *some* relation R.

2) $R_1 R_2 \neq R, \breve{R}$. $R_1 = R_2$ or $\breve{R}_2$. If $R_1 = R_2$, $\alpha R_1 \xi$ and $\xi R_1 \beta$. Let $\alpha R_1 \eta$ and $\xi R_1 \eta$; then η is on $\alpha\xi$, $\xi\beta$ and $\eta \neq \xi$, which is impossible. If $R_1 = \breve{R}_2$, then we have $\alpha R_1 \xi$ and $\beta R_1 \xi$; that is, β is on $\alpha\xi$.

Other cases differ from the preceding only formally, so that we may say, if three points α, β, ξ form a descriptive triangle, then the corresponding relations $R, R_1, R_2, \breve{R}, \breve{R}_1, \breve{R}_2$ are distinct from each other. Further, on the basis of the preceding we may say that if two descriptive lines intersect, then the corresponding relations

$$R_1, \quad R_2, \quad \breve{R}_1, \quad \breve{R}_2$$

are distinct. To construct, therefore, an n-dimensional descriptive geometry $(n > 1)$ an infinitude of relations of the type R is required.* However, all the axioms of system ${}^1R_1$ except axiom 4 are satisfied if we define $\alpha R \beta$ and $\alpha' R \beta'$ to mean that $\alpha\beta$ and $\alpha'\beta'$ are two euclidian parallel and similarly directed† segments. Thus any set of euclidian, similarly directed parallels in n-space $(n > 1)$ will satisfy all the axioms of the system ${}^1R_1$ except axiom 4; so that, on the basis of the remaining axioms, the single relation R is capable of generating an (unlimited) linear vector.

The independence of the axioms of the system ${}^1R_1$ may be briefly discussed. Axiom 1 is obviously independent, since if there is no point, the remaining axioms are not effective. For axiom 2, take one point α such that $\alpha \bar{R} \alpha$. For axiom 3, take one point α such that $\alpha R \alpha$. The independence of axiom 4 is established by the above finite system of 21 dyads; the latter system also shows that axiom 4 is necessary to prove the existence of an infinitude of points. To show that axiom 5 is independent, take as points the ordinary system of rational numbers, and the imaginary unit i ordered thus: $\alpha R \beta$ means $\alpha < \beta$, and we define $i R \beta$ if $\beta > 0$ and $\alpha R i$ if $\alpha < 0$; also $0 \bar{R} i$, $i \bar{R} 0$. Then axiom 5 is contradicted, since $(-1) R 0$ and $(-1) R i$ do not imply $i R 0$ or $0 R i$; the remaining axioms are satisfied. The independence systems for axioms 6 and 7 are the sets of positive and negative rational numbers, respectively; the independence system for axiom 8 consists of the positive and negative integers ordered in the usual manner.

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* In the extended system, instead of axiom 4, we have the axiom of transitiveness: $\alpha R \xi$ and $\xi R \beta$ imply $\alpha R \beta$.

† Not to be confounded with sameness of sense. This confusion has occurred with many authors. Similarity of direction is excluded by means of the axiom, "$\alpha R \beta$ and $\gamma R \delta$ imply $\alpha R \gamma$ or $\gamma R \alpha$"; cf. our paper, *Trans. Am. Math. Soc.*, July (1909), p. 309.

System ${}^2R_2$. I. *Axioms.*

1. There exists α.

2. The existence of α implies the existence of $\alpha_0, \beta_0, \gamma_0$ such that $\alpha_0 R \beta_0 \gamma_0$ or $\alpha_0 R \gamma_0 \beta_0$.

3. $\alpha R \beta \gamma$ implies $\alpha \bar{R} \gamma \beta$.

4. $\alpha R \beta \gamma$ implies $\beta R \gamma \alpha$.

5. $\alpha R \beta \gamma$ and $\xi \neq \alpha, \beta, \gamma$ imply $\xi R \beta \gamma$ or $\alpha R \xi \gamma$ or $\alpha R \beta \xi$.

6. $\alpha R \beta \gamma$ and $\xi R \beta \gamma$ and $\xi \neq \alpha$ imply $\xi R \gamma \alpha$ or $\xi R \alpha \gamma$ or $\xi R \beta \alpha$ or $\xi R \alpha \beta$.

7. $\alpha R \beta \gamma$ and $\alpha \neq \beta \neq \gamma \neq \alpha$* imply the existence of δ such that $\delta R \alpha \gamma$ and $\delta R \beta \alpha$.

8. $\alpha R \beta \gamma$, $\varepsilon R \beta \gamma$, $\alpha R \varepsilon \gamma$, $\alpha R \beta \varepsilon$, $\varepsilon \neq \alpha, \beta, \gamma$ imply: the existence of ξ such that $\xi \bar{R} \alpha \varepsilon$, $\xi \bar{R} \varepsilon \alpha$, and the existence of δ' such that $\delta' R \beta \gamma$ implies $\delta' R \xi \gamma$ and $\delta' R \beta \xi$, and the existence of δ'' such that $\delta'' R \gamma \beta$ implies $\delta'' R \xi \beta$ and $\delta'' R \gamma \xi$.

II. *Definitions.*

1. ξ is on $\alpha \beta \gamma$ means, $\alpha R \beta \gamma$ and ($\xi R \beta \gamma$ or $\alpha R \xi \gamma$ or $\alpha R \beta \xi$).

2. ξ is on $\alpha \beta$ means, $\alpha \neq \beta$ and the existence of γ such that $\gamma R \alpha \beta$ implies $\gamma R \alpha \xi$ or $\gamma R \xi \beta$. (Cf. Fig. 1; shaded portion, including linear boundary through α, β, indicates possible domain of ξ.)

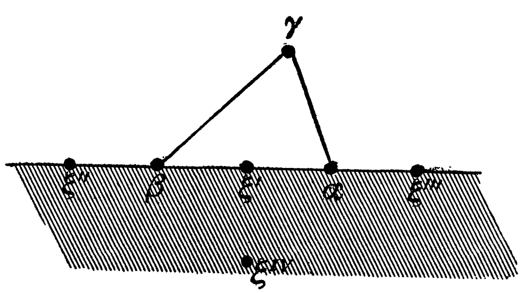

FIG. 1.

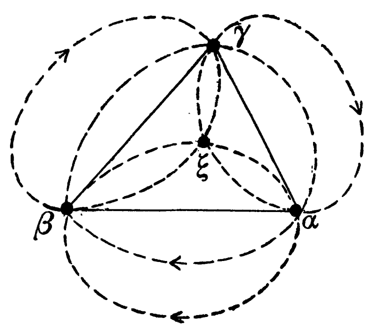

FIG. 2.

3. ξ is in the interior of $\alpha \beta \gamma$ is defined by substituting, in definition 1, "and" for "or". (Cf. Fig. 2.)

4. ξ is in the interior of $\alpha \beta$ is defined by substituting, in definition 2, "and" for "or". (Cf. Figs. 3.)

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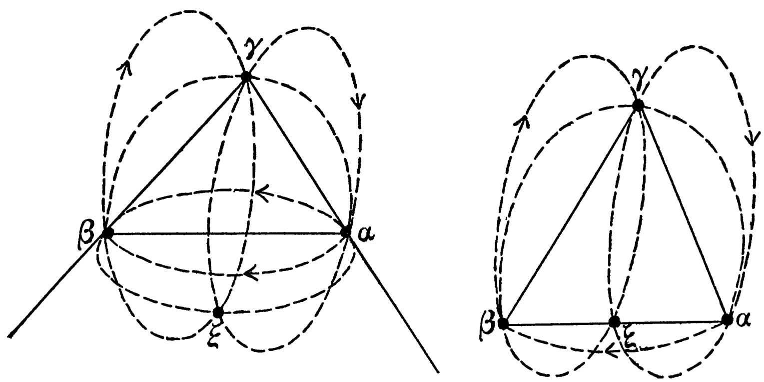
* That is, α, β, γ are distinct.

As an equivalent definition under the axioms we may give:

ξ is on $\alpha\beta$ means, $\alpha \neq \beta$ and $\xi \bar{R} \alpha\beta$.

For if $\alpha \neq \beta$ and $\xi \bar{R} \alpha\beta$, the existence of γ such that $\gamma R \alpha\beta$ implies $\gamma R \alpha\xi$ or $\gamma R \xi\beta$, by axiom 5. Conversely, if the existence of γ such that $\gamma R \alpha\beta$ implies $\gamma R \alpha\xi$ or $\gamma R \xi\beta$, then $\xi \bar{R} \alpha\beta$; for $\xi R \alpha\beta$ implies $\xi R \alpha\xi$ or $\xi R \xi\beta$, which in either case is impossible under the axioms.



FIGS. 3.

If we adopt the latter definition, then the corresponding definition of "ξ in the interior of $\alpha\beta$" is:

ξ is in the interior of $\alpha\beta$ means, $\alpha \neq \beta$, $\xi \bar{R} \alpha\beta$ and the existence of γ such that $\gamma R \alpha\beta$ and ($\gamma R \xi\beta$ or $\gamma R \alpha\xi$) implies $\gamma R \xi\beta$ and $\gamma R \alpha\xi$.

On the basis of the above definitions we may define:

ξ is on the plane of the points α, β, γ means, ξ is on $\alpha\beta\gamma$ or $\beta\alpha\gamma$.

ξ is on the line of the points α, β means, ξ is on $\alpha\beta$ and $\beta\alpha$.

ξ is in the interior of, or between,* the points α, β, γ means, ξ is in the interior of $\alpha\beta\gamma$ or $\beta\alpha\gamma$.

ξ is in the interior of, or between, the points α, β means, ξ is in the interior of $\alpha\beta$ and $\beta\alpha$.

III. *Theorems.*

1. $\alpha R \beta\gamma$ implies $\alpha \neq \beta \neq \gamma \neq \alpha$.

By axiom 4, $\alpha R \beta\gamma$ implies $\beta R \gamma\alpha$; similarly, $\beta R \gamma\alpha$ implies $\gamma R \alpha\beta$. Hence, by axiom 3, we have, respectively, $\beta \neq \gamma$, $\gamma \neq \alpha$, $\alpha \neq \beta$; *i. e.*, $\alpha \neq \beta \neq \gamma \neq \alpha$.

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* On the terminology cf. Grassmann, "Ausdehnungslehre," 1844, § 110 (Eckgebilde); also Vahlen, "Abstrakte Geometrie," p. 10.

2. There exist three distinct points.

By axioms 1 and 2, and theorem 1.

3. $\xi R \beta\gamma$, $\alpha R \xi\gamma$, $\alpha R \beta\xi$ imply $\alpha R \beta\gamma$.

We have $\xi R \beta\gamma$ and $\alpha \neq \xi, \beta, \gamma$. Hence by axiom 5, $\alpha R \beta\gamma$ or $\xi R \alpha\gamma$ or $\xi R \beta\alpha$. But from the hypothesis it follows that $\xi R \gamma\alpha$ and $\xi R \alpha\beta$. Hence by axiom 3 we must have $\alpha R \beta\gamma$.

4. $\alpha R \beta\gamma$ and $\alpha R \xi\gamma$, $\xi \neq \beta$ imply $\xi R \beta\gamma$ or $\xi R \gamma\beta$ or $\xi R \alpha\beta$ or $\xi R \beta\alpha$.

This follows at once from axiom 6, since we have $\beta R \gamma\alpha$ and $\xi R \gamma\alpha$ by axiom 4.

5. $\xi \bar{R} \alpha\beta$, $\xi \bar{R} \beta\alpha$, $\eta \bar{R} \alpha\beta$, $\eta \bar{R} \beta\alpha$, $\xi \neq \eta$, $\alpha \neq \beta$ imply $\alpha \bar{R} \xi\eta$, $\alpha \bar{R} \eta\xi$, $\beta \bar{R} \xi\eta$, $\beta \bar{R} \eta\xi$.

It is sufficient to prove that $\alpha \bar{R} \xi\eta$, $\alpha \bar{R} \eta\xi$. Suppose $\alpha R \xi\eta$. Then since $\beta \neq \alpha$ and we may assume $\beta \neq \xi, \eta$, we have by axiom 5, $\beta R \xi\eta$ or $\alpha R \beta\eta$ or $\alpha R \xi\beta$. Therefore $\beta R \xi\eta$. But by axiom 6, $\alpha R \xi\eta$ and $\beta R \xi\eta$ imply $\alpha R \beta\eta$ or $\alpha R \eta\beta$ or $\alpha R \xi\beta$ or $\alpha R \beta\xi$. Hence $\beta R \xi\eta$ is impossible; *i. e.*, $\alpha \bar{R} \xi\eta$ is true. Similarly, $\alpha \bar{R} \eta\xi$.

6. $\xi \neq \eta$ implies the existence of ζ such that $\zeta R \xi\eta$.

By axioms 1, 2 there exist the points $\alpha_0, \beta_0, \gamma_0$ such that, say, $\alpha_0 R \beta_0\gamma_0$. If $\xi \neq \alpha_0, \beta_0, \gamma_0$, we have by axiom 5, $\xi R \beta_0\gamma_0$ or $\alpha_0 R \xi\gamma_0$ or $\alpha_0 R \beta_0\xi$. Let $\xi R \beta_0\gamma_0$. Then if $\eta \neq \xi, \beta_0, \gamma_0$ by axiom 5, $\eta R \beta_0\gamma_0$ or $\xi R \eta\gamma_0$ or $\xi R \beta_0\eta$. If $\eta R \beta_0\gamma_0$, since $\xi R \beta_0\gamma_0$ we have by axiom 6, $\xi R \eta\gamma_0$ or $\xi R \gamma_0\eta$ or $\xi R \beta_0\eta$ or $\xi R \eta\beta_0$. Suppose $\xi R \gamma_0\eta$; that is, $\gamma_0 R \eta\xi$. Then by axiom 7 there exists a point ζ such that $\zeta R \xi\eta$.

In the preceding proof is contained implicitly the following theorem:

7. $\alpha R \beta\gamma$, $\xi \neq \eta$ imply $\xi R \eta\alpha$ or $\eta R \xi\alpha$ or $\xi R \eta\beta$ or $\eta R \xi\beta$ or $\xi R \eta\gamma$ or $\eta R \xi\gamma$.

8. $\alpha \neq \beta$ implies the existence of ξ such that ξ is in the interior of α, β.

By theorem 6, $\alpha \neq \beta$ implies the existence of γ such that $\gamma R \alpha\beta$. Then by axiom 7 there is a δ such that $\delta R \gamma\beta$ and $\delta R \alpha\gamma$. Since $\gamma R \alpha\beta$, $\delta R \gamma\beta$ and $\delta R \alpha\gamma$, we have by theorem 3, $\delta R \alpha\beta$. Therefore, by axiom 8, there is a point ξ in the interior of $\alpha\beta$ and $\beta\alpha$.

9. $\alpha R \beta\gamma$, $\alpha' R \gamma\beta$ and $\alpha \neq \alpha'$ imply the (unique) existence of ξ such that $\xi \bar{R} \beta\gamma$, $\xi \bar{R} \gamma\beta$ and ξ is in the interior of α', α.

Since $\beta \neq \gamma$, there exists by theorem 8 a point ξ_0 in the interior of β, γ. Then let, first, $\xi_0 R \alpha\alpha'$ or $\xi_0 R \alpha'\alpha$. Suppose $\xi_0 R \alpha\alpha'$. Since ξ_0 is in the interior of β, γ, $\xi_0 R \gamma\alpha$ and $\xi_0 R \alpha'\gamma$; that is, ξ_0 is in the interior of $\alpha\alpha'\gamma$.

Hence by axiom 8 there exists an η such that η is in the interior of α, α' and $\eta \bar{R} \xi_0 \gamma$, $\eta \bar{R} \gamma \xi_0$. Since $\xi_0 \bar{R} \beta \gamma$ and $\xi_0 \bar{R} \gamma \beta$, it follows by theorem 5 that $\eta \bar{R} \beta \gamma$, $\eta \bar{R} \gamma \beta$. A similar proof holds if $\xi_0 R \alpha' \alpha$.

Let now $\xi_0 \bar{R} \alpha \alpha'$, $\xi_0 \bar{R} \alpha' \alpha$. Since $\xi_0 \neq \gamma$, there is an η in the interior of ξ_0, γ, by theorem 8. Hence $\eta R \gamma \alpha$, $\eta R \alpha \xi$ and $\eta R \alpha' \gamma$ and $\eta R \xi \alpha'$. Since $\eta R \xi \alpha'$ and $\alpha \neq \eta$, ξ, α' by axiom 5, $\alpha R \xi \alpha'$ or $\eta R \alpha \alpha'$ or $\eta R \xi \alpha$. Hence $\eta R \alpha \alpha'$. That is, η is in the interior of $\alpha \alpha' \gamma$. Hence by axiom 8 there is a point ζ in the interior of α, α' and such that $\zeta \bar{R} \gamma \eta$, $\zeta \bar{R} \eta \gamma$. Since $\eta \bar{R} \xi_0 \gamma$, $\eta \bar{R} \gamma \xi_0$ and $\xi_0 \bar{R} \beta \gamma$, $\xi_0 \bar{R} \gamma \beta$, it follows, by theorem 5, $\zeta \bar{R} \beta \gamma$, $\zeta \bar{R} \gamma \beta$. Thus the theorem is valid.

With the aid of the preceding theorems we can easily derive, on the basis of the axioms 1–8, the usual properties of the (unique) descriptive plane,* excepting, of course, the continuous property. A set of points satisfying system ${}^2R_2$ may be said to be cyclically ordered.

We inquire now into the extensibility of the system ${}^2R_2$. In the extended system, the axiom of dimensionality, axiom 5, will be contradicted, the remaining axioms being valid. There exists a finite system of planar triads such that axioms 1–4, 6–8 are satisfied or are not effective, and axiom 5 is contradicted; it is the system consisting of an arbitrary number of closed chains of the type

$$[\xi \beta \gamma] \qquad [\alpha \xi \gamma] \qquad [\alpha \beta \xi] \qquad [\beta \alpha \gamma].$$

In such a system, axioms 1–4 are satisfied; axiom 5 is contradicted; axiom 6 is not effective; axiom 7 is satisfied; axiom 8 is not effective. The preceding system of triads is, however, not a part of a descriptive geometry; for if so, there would be a point θ such that $\theta R \xi \beta$ and in the plane $\alpha \beta \xi$. Then θ is in the planes $\alpha \beta \xi$, $\xi \beta \gamma$ and α is not in the plane $\xi \beta \gamma$. But this is impossible in a descriptive geometry. Let us require, then, that the extended system be descriptive. Let ξ be not on the plane $\alpha \beta \gamma$. That is, if $\alpha R \beta \gamma$ we have $\xi \bar{R} \beta \gamma$, $\alpha \bar{R} \xi \gamma$, $\alpha \bar{R} \beta \xi$. We may suppose $\alpha R \beta \gamma$, $\xi R_1 \beta \gamma$, $\xi R_2 \gamma \alpha$, $\xi R_3 \alpha \beta$. We consider these cases:

1) $R = R_1$ or R_2 or R_3. If $R = R_1$ ξ is on $\alpha \beta \gamma$, which is contrary to hypothesis. Similarly, $R \neq R_2$, R_3.

2) $\breve{R} = R_1$ or R_2 or R_3, where $\breve{R}$ is the conjugate of R. If $\breve{R} = R_1$, we have $\alpha R \beta \gamma$ and $\xi R \gamma \beta$. Let θ be in the plane $\xi \gamma \beta$ and such that $\theta R \beta \gamma$.

* Cf. E. H. Moore, *Trans. Am. Math. Soc.*, Vol. III (1902), p. 142.

Then θ is in the planes $\alpha\beta\gamma$, $\xi\gamma\beta$, not on the line β,γ, and ξ is not on the plane $\alpha\beta\gamma$. Hence $\breve{R}=R_1$ is impossible; similarly, $\breve{R}=R_2$ or R_3 is impossible.

3) R_1, R_2, $R_3 \neq R$, $\breve{R}$. Suppose $R_1=R_2$. Then $\xi R_1\beta\gamma$ and $\xi R_1\gamma\alpha$, *i. e.*, $\beta R_1\gamma\xi$, $\alpha R_1\xi\gamma$. Since ξ is not on the plane $\alpha\beta\gamma$, α is not on the plane $\beta\gamma\xi$. Then we may take a point θ in the plane $\alpha\xi\gamma$ such that $\theta R_1\gamma\xi$. Then θ is also in the plane $\beta\gamma\xi$. Since θ is not on the line γ, ξ, $R_1=R_2$ is impossible. Let $\breve{R}_1=R_2$. Then $\xi R_1\beta\gamma$ and $\xi R_1\alpha\gamma$. Hence α is on $\xi\beta\gamma$. That is, ξ is on $\alpha\beta\gamma$. Therefore, $\breve{R}_1=R_2$ is impossible.

Since the preceding cases are the only essential ones that can arise, we conclude: if ξ is without the plane $\alpha\beta\gamma$, then the corresponding relations

$$R,\ R_1,\ R_2,\ R_3,\ \breve{R},\ \breve{R}_1,\ \breve{R}_2,\ \breve{R}_3$$

are distinct. We may say further, on the basis of the preceding, that if two descriptive planes intersect, the corresponding relations

$$R',\ R'',\ \breve{R}',\ \breve{R}''$$

are distinct. To construct, therefore, an n-dimensional descriptive geometry ($n>2$) an infinitude of planar relations of the type R is required. However, axioms 1–4, 6–8 are satisfied if we make the agreement that $\alpha R\beta\gamma$ and $\alpha' R\beta'\gamma'$ mean that $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ are two euclidean parallel and similarly directed triangular segments; that is, any class of euclidean parallel, similarly directed planes satisfies the above-mentioned axioms; the relation R, then, may be said to be capable of generating in this case an (unlimited) planar vector.

The independence of axioms 1–8 can be established in the following manner. Let us denote by C_n ($n=1,2,3,\ldots,8$) the class of points such that with respect to this class axiom n is contradicted and the remaining axioms are satisfied or are not effective. Then C_1 consists of no point. C_2 consists of one point α such that $\alpha\bar{R}\alpha\alpha$. C_3 consists of one point α such that $\alpha R\alpha\alpha$. C_4 consists of two points α,β such that $\alpha R\alpha\beta$ and $\alpha\bar{R}\beta\alpha$. C_5 is indicated above. For C_6 we take the ordinary Cartesian plane (rational coordinates) and the point $\iota=(i_1 i_2)$, where ι is distinct from the elements of the preceding plane. The class is ordered thus: the plane is ordered in the usual manner; $\iota R\beta\gamma$ is valid if, and only if, $\omega R\beta\gamma$, where $\omega=(0,0)$; and if ξ is any point, then $\xi\bar{R}\omega\iota$, $\xi\bar{R}\iota\omega$. The necessary and sufficient condition that $\alpha R\beta\gamma$ is

$$\begin{vmatrix} \alpha_1\,\alpha_2\,1 \\ \beta_1\,\beta_2\,1 \\ \gamma_1\,\gamma_2\,1 \end{vmatrix} > 0,$$

where $\alpha = (\alpha_1\alpha_2)$, $\beta = (\beta_1\beta_2)$, $\gamma = (\gamma_1\gamma_2)$. Thus if $\alpha = \iota$, $\beta_1\gamma_2 - \beta_2\gamma_1 > 0$.

C_7 consists of three points α, β, γ with the agreement that $\alpha R \beta \gamma$.

C_8 is the set of points in the usual Cartesian plane with integral coordinates; $\alpha R \beta \gamma$ if, and only if,

$$\begin{vmatrix} \alpha_1\,\alpha_2\,1 \\ \beta_1\,\beta_2\,1 \\ \gamma_1\,\gamma_2\,1 \end{vmatrix} > 0,$$

where $\alpha = (\alpha_1\alpha_2)$, $\beta = (\beta_1\beta_2)$, $\gamma = (\gamma_1\gamma_2)$.

Thus it will be observed that axioms 1–5, 7 are necessary to prove the existence of an infinitude of points.

[We note that for axioms 3, 4, 5 of ${}^2R_2$ the following axioms may be substituted:

3'. $\alpha R \beta \gamma$ implies $\beta \bar{R} \alpha \gamma$.

4'. $\alpha R \beta \gamma$ implies $\gamma \bar{R} \beta \alpha$.

5'. $\alpha R \beta \gamma$, $\xi \neq \alpha$ imply $\xi R \beta \gamma$ or $\xi R \gamma \alpha$ or $\xi R \alpha \beta$.

Theorem. $\alpha R \beta \gamma$ implies $\alpha \bar{R} \gamma \beta$.

By axiom 3', $\beta \neq \alpha$; hence if $\alpha R \gamma \beta$, by axiom 5', $\beta R \gamma \beta$ or $\beta R \beta \alpha$ or $\beta R \alpha \gamma$. By axiom 4', $\beta R \gamma \beta$ is impossible; $\beta R \beta \alpha$ and $\beta R \alpha \gamma$ are impossible by axiom 3' and the hypothesis. Hence $\alpha R \gamma \beta$ is impossible.

Theorem. $\alpha R \beta \gamma$ implies $\beta R \gamma \alpha$.

By axiom 3', $\alpha R \beta \gamma$ implies $\beta \neq \alpha$. Hence by axiom 5', $\beta R \beta \gamma$ or $\beta R \gamma \alpha$ or $\beta R \alpha \beta$. By axioms 3' and 4', $\beta R \beta \gamma$ and $\beta R \alpha \beta$ are impossible; hence $\beta R \gamma \alpha$.

Theorem. $\alpha R \beta \gamma$, $\xi \neq \alpha, \beta, \gamma$ imply $\xi R \beta \gamma$ or $\alpha R \xi \gamma$ or $\alpha R \beta \xi$.

Proof follows at once from axiom 5' through the previous theorem.]

System ${}^3R_3$. I. *Axioms.*

1. There exists α.

2. The existence of α implies the existence of $\alpha_0 \beta_0 \gamma_0 \delta_0$ such that $\alpha_0 R \beta_0 \gamma_0 \delta_0$ or $\alpha_0 R \beta_0 \delta_0 \gamma_0$.

3. $\alpha R \beta \gamma \delta$ implies $\alpha \bar{R} \beta \delta \gamma$.

4. $\alpha R \beta \gamma \delta$ implies $\alpha R \gamma \delta \beta$.

5. $\alpha R \beta \gamma \delta$ implies $\gamma R \delta \alpha \beta$.

6. $\alpha R \beta \gamma \delta$, $\xi \neq \alpha, \beta, \gamma, \delta$ imply $\xi R \beta \gamma \delta$ or $\alpha R \xi \gamma \delta$ or $\alpha R \beta \xi \delta$ or $\alpha R \beta \gamma \xi$.

7. $\alpha R \beta \gamma \delta$ and $\xi R \beta \gamma \delta$, $\xi \neq \alpha$ imply $\alpha R \xi \gamma \delta$ or $\alpha R \xi \delta \gamma$ or $\alpha R \beta \xi \delta$ or $\alpha R \beta \delta \xi$ or $\alpha R \beta \gamma \xi$ or $\alpha R \beta \xi \gamma$.

8. $\alpha R \beta \gamma \delta$, $\xi R \beta \gamma \eta$, $\xi \neq \alpha, \delta$, $\eta \neq \alpha, \delta$ imply $\xi R \beta \gamma \delta$ or $\xi R \beta \delta \gamma$ or $\xi R \beta \gamma \alpha$ or $\xi R \beta \alpha \gamma$.

9. $\alpha R \beta \gamma \delta$, $\alpha \neq \beta \neq \gamma \neq \delta \neq \alpha$ imply the existence of ε such that $\varepsilon R \alpha \gamma \delta$, $\varepsilon R \beta \alpha \delta$, $\varepsilon R \beta \gamma \alpha$.

10. $\alpha R \beta \gamma \delta$, $\varepsilon R \beta \gamma \delta$, $\alpha R \varepsilon \gamma \delta$, $\alpha R \beta \varepsilon \delta$, $\alpha R \beta \gamma \varepsilon$, $\varepsilon \neq \alpha, \beta, \gamma, \delta$ imply: the existence of ξ such that $\xi \bar{R} \alpha \beta \varepsilon$, $\xi \bar{R} \alpha \varepsilon \beta$, and the existence of $\delta' \delta''$ such that $\delta' R \delta'' \gamma \delta$ implies $\delta' R \delta'' \xi \delta$ and $\delta' R \delta'' \gamma \xi$.

II. *Definitions.*

1. ξ is on $\alpha \beta \gamma \delta$ means, $\alpha R \beta \gamma \delta$ and ($\xi R \beta \gamma \delta$ or $\alpha R \xi \gamma \delta$ or $\alpha R \beta \xi \delta$ or $\alpha R \beta \gamma \xi$).

2. ξ is on $\alpha \beta \gamma$ means, There exists δ' such that $\delta' R \alpha \beta \gamma$, and the existence of δ such that $\delta R \alpha \beta \gamma$ implies $\delta R \xi \beta \gamma$ or $\delta R \alpha \xi \gamma$ or $\delta R \alpha \beta \xi$.

3. ξ is on $\alpha \beta$ means, $\alpha \neq \beta$ and the existence of $\delta' \delta''$ such that $\delta' R \delta'' \alpha \beta$ implies $\delta' R \delta'' \xi \beta$ or $\delta' R \delta'' \alpha \xi$.

Definitions for ξ in the interior of $\alpha \beta \gamma \delta$, $\alpha \beta \gamma$, $\alpha \beta$ are obtained from the above by substituting for "or", "and". We notice that if ξ is on $\alpha \beta$, ξ is also on $\beta \alpha$; but if ξ is on $\alpha \beta \gamma$, ξ is not necessarily on $\alpha \gamma \beta$.

As equivalent definitions under the axioms we may give:

ξ is on $\alpha \beta \gamma$ means, There exists δ' such that $\delta' R \alpha \beta \gamma$ and $\xi \bar{R} \alpha \beta \gamma$.

ξ is on $\alpha \beta$ means, $\alpha \neq \beta$ and the existence of δ implies $\delta \bar{R} \alpha \beta \xi$.

If we adopt the latter definitions, then we define correspondingly:

ξ is in the interior of $\alpha \beta \gamma$ means, There exists a δ' such that $\delta' R \alpha \beta \gamma$, and the existence of δ such that $\delta R \alpha \beta \gamma$ and ($\delta R \xi \beta \gamma$ or $\delta R \alpha \xi \gamma$ or $\delta R \alpha \beta \xi$) imply $\delta R \xi \beta \gamma$, $\delta R \alpha \xi \gamma$, $\delta R \alpha \beta \xi$.

ξ is in the interior of $\alpha \beta$ means, $\alpha \neq \beta$; the existence of δ implies $\delta \bar{R} \alpha \beta \xi$ and the existence of $\delta' \delta''$ such that $\delta' R \delta'' \alpha \beta$ and ($\delta' R \delta'' \xi \beta$ or $\delta' R \delta'' \alpha \xi$) imply $\delta' R \delta'' \xi \beta$, $\delta' R \delta'' \alpha \xi$.

On the basis of the above definitions we may define:

ξ is in the space of the points $\alpha, \beta, \gamma, \delta$ means, ξ is on $\alpha \beta \gamma \delta$ or $\alpha \beta \delta \gamma$.

ξ is on the plane of the points α, β, γ means, ξ is on $\alpha \beta \gamma$ and $\alpha \gamma \beta$.

ξ is on the line of the points α, β means, ξ is on $\alpha \beta$.

ξ is in the interior of, or between, the points $\alpha, \beta, \gamma, \delta$ means, ξ is in the interior of $\alpha\beta\gamma\delta$ or $\alpha\beta\delta\gamma$.

ξ is in the interior of, or between, the points α, β, γ means, ξ is in the interior of $\alpha\beta\gamma$ and $\alpha\gamma\beta$.

ξ is in the interior of, or between, the points α, β means, ξ is in the interior of $\alpha\beta$.

III. *Theorems.*

In Chapter V we indicate how the system ${}^3R_3$ implies the system ${}^3K_3$. In that chapter we also show that system ${}^3K_3$ is sufficient for projective geometry if an axiom of continuity is added; hence a similar theorem is valid of system ${}^3R_3$. A set of points satisfying axioms 1–10 is said to be spherically ordered.*

As to the extensibility of the system ${}^3R_3$, we remark first that there exists a finite system of three-dimensional tetrads such that axioms 1–5, 7–10 are satisfied or are not effective, and axiom 6 is contradicted; it is the system consisting of an arbitrary number of closed three-dimensional chains of the type

$$[\xi\beta\gamma\delta] \quad [\alpha\xi\gamma\delta] \quad [\alpha\beta\xi\delta] \quad [\alpha\beta\gamma\xi] \quad [\beta\alpha\gamma\delta].$$

In such a system, axioms 1–5 are satisfied, axiom 6 is contradicted, axioms 7, 8 are not effective, axiom 9 is satisfied, axiom 10 is not effective. The preceding system of tetrads is, however, not a part of a descriptive geometry. If we require that the extended system have this property, then in a manner analogous to that indicated under system ${}^2R_2$, it may be shown that if two descriptive three-dimensional spaces intersect, the corresponding relations

$$R', \quad R'', \quad \breve{R}', \quad \breve{R}''$$

are distinct. To construct, therefore, an n-dimensional descriptive geometry ($n > 3$) an infinitude of three-dimensional relations of the type R is required. However, axioms 1–5, 7–10 are satisfied if we make the agreement that $\alpha R \beta\gamma\delta$, $\alpha' R \beta'\gamma'\delta'$ mean that $\alpha\beta\gamma\delta$, $\alpha'\beta'\beta'\delta'$ are two euclidian parallel and similarly directed tetrahedral segments; that is, any class of euclidian parallel, similarly directed three-dimensional spaces satisfy the above-mentioned axioms; the relation R, then, may be said to be capable of generating in this case a three-dimensional (unlimited) vector.

Axioms 1–6, 9 can be proved independent by means of finite classes. For C_1 consists of no point; C_2 consists of one point α such that $\alpha\bar{R}\alpha\alpha\alpha$; C_3 consists

* Compare Vahlen, "Abstrakte Geometrie," p. 10.

of the point α such that $\alpha R \alpha\alpha\alpha$; C_4 consists of two points α, β such that $\alpha R \beta\beta\alpha$, $\beta R \alpha\alpha\beta$, $\alpha \bar{R} \beta\alpha\beta$, $\beta \bar{R} \alpha\beta\alpha$; C_5 consists of three points α, β, γ such that $\alpha R \alpha\beta\gamma$, $\alpha R \beta\gamma\alpha$, $\alpha R \gamma\alpha\beta$, $\beta \bar{R} \gamma\alpha\alpha$, $\gamma \bar{R} \alpha\alpha\beta$, $\alpha \bar{R} \alpha\gamma\beta$; C_6 is indicated above; C_9 consists of four points $\alpha, \beta, \gamma, \delta$ such that $\alpha R \beta\gamma\delta$. Thus axioms 1–6, 9 are necessary for an infinitude of points.

[We note that for axioms 3, 4, 5, 6 the following axioms may be substituted:

3′. $\alpha R \beta\gamma\delta$ implies $\beta \bar{R} \alpha\gamma\delta$.

4′. $\alpha R \beta\gamma\delta$ implies $\gamma \bar{R} \beta\alpha\delta$.

5′. $\alpha R \beta\gamma\delta$ implies $\delta \bar{R} \beta\gamma\alpha$.

6′. $\alpha R \beta\gamma\delta$, $\xi \neq \alpha$ imply $\xi R \beta\gamma\delta$ or $\xi R \gamma\alpha\delta$ or $\xi R \delta\alpha\beta$ or $\xi R \alpha\gamma\beta$.

Theorem. $\alpha R \beta\gamma\delta$ implies $\beta R \gamma\alpha\delta$.

Since $\alpha R \beta\gamma\delta$, by 3′, $\beta \neq \alpha$; hence by 6′, $\beta R \beta\gamma\delta$ or $\beta R \gamma\alpha\delta$ or $\beta R \delta\alpha\beta$ or $\beta R \alpha\gamma\beta$. By 3′, $\beta \bar{R} \beta\gamma\delta$; by 5′, $\beta \bar{R} \delta\alpha\beta$ and $\beta \bar{R} \alpha\gamma\beta$.

Theorem. $\alpha R \beta\gamma\delta$ implies $\gamma R \delta\alpha\beta$.

Since $\alpha R \beta\gamma\delta$, by 4′, $\gamma \neq \alpha$; hence by 6′, $\gamma R \beta\gamma\delta$ or $\gamma R \gamma\alpha\delta$ or $\gamma R \delta\alpha\beta$ or $\gamma R \alpha\gamma\beta$. By 4′, $\gamma \bar{R} \beta\gamma\delta$; by 3′, $\gamma \bar{R} \gamma\alpha\delta$; by 4′, $\gamma \bar{R} \alpha\gamma\beta$; hence $\gamma R \delta\alpha\beta$.

Theorem. $\alpha R \beta\gamma\delta$ implies $\alpha \bar{R} \beta\delta\gamma$.

Proof by 3′–5′ and the preceding theorems.

Theorem. $\alpha R \beta\gamma\delta$, $\xi \neq \alpha, \beta, \gamma, \delta$ imply $\xi R \beta\gamma\delta$ or $\alpha R \xi\gamma\delta$ or $\alpha R \beta\xi\delta$ or $\alpha R \beta\gamma\xi$.

Proof follows at once from 6′ by the first two theorems.]

System ${}^4R_4$. I. *Axioms.*

1. There exists α.

2. The existence of α implies the existence of $\alpha_1^0, \alpha_2^0, \alpha_3^0, \alpha_4^0, \alpha_5^0$ such that $\alpha_1^0 R \alpha_2^0\alpha_3^0\alpha_4^0\alpha_5^0$ or $\alpha_1^0 R \alpha_2^0\alpha_3^0\alpha_5^0\alpha_4^0$.

3. $\alpha_1 R \alpha_2\alpha_3\alpha_4\alpha_5$ implies $\alpha_1 \bar{R} \alpha_2\alpha_3\alpha_5\alpha_4$.

4. $\alpha_1 R \alpha_2\alpha_3\alpha_4\alpha_5$ implies $\alpha_2 R \alpha_3\alpha_1\alpha_4\alpha_5$.

5. $\alpha_1 R \alpha_2\alpha_3\alpha_4\alpha_5$ implies $\alpha_3 R \alpha_4\alpha_1\alpha_2\alpha_5$.

6. $\alpha_1 R \alpha_2\alpha_3\alpha_4\alpha_5$ implies $\alpha_4 R \alpha_5\alpha_1\alpha_2\alpha_3$.

7. $\alpha_1 R \alpha_2\alpha_3\alpha_4\alpha_5$, $\xi \neq \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ imply $\xi R \alpha_2\alpha_3\alpha_4\alpha_5$ or $\alpha_1 R \xi\alpha_3\alpha_4\alpha_5$ or $\alpha_1 R \alpha_2\xi\alpha_4\alpha_5$ or $\alpha_1 R \alpha_2\alpha_3\xi\alpha_5$ or $\alpha_1 R \alpha_2\alpha_3\alpha_4\xi$.

8. $\alpha_1 R \alpha_2\alpha_3\alpha_4\alpha_5$, $\xi R \alpha_2\alpha_3\alpha_4\alpha_5$, $\xi \neq \alpha_1$ imply $\alpha_1 R \xi\alpha_3\alpha_4\alpha_5$ or $\alpha_1 R \xi\alpha_3\alpha_5\alpha_4$ or $\alpha_1 R \alpha_2\xi\alpha_4\alpha_5$ or $\alpha_1 R \alpha_2\xi\alpha_5\alpha_4$ or $\alpha_1 R \alpha_2\alpha_3\xi\alpha_5$ or $\alpha_1 R \alpha_2\alpha_3\alpha_5\xi$ or $\alpha_1 R \alpha_2\alpha_3\alpha_4\xi$ or $\alpha_1 R \alpha_2\alpha_3\xi\alpha_4$.

9. $\alpha_1 R \alpha_2 \alpha_3 \alpha_4 \alpha_5$, $\xi R \alpha_2 \alpha_3 \alpha_4 \eta_1$, $\xi \neq \alpha_1, \alpha_5$, $\eta_1 \neq \alpha_1, \alpha_5$ imply $\xi R \alpha_2 \alpha_3 \alpha_4 \alpha_5$ or $\xi R \alpha_2 \alpha_3 \alpha_5 \alpha_4$ or $\xi R \alpha_2 \alpha_3 \alpha_4 \alpha_1$ or $\xi R \alpha_2 \alpha_3 \alpha_1 \alpha_4$.

10. $\alpha_1 R \alpha_2 \alpha_3 \alpha_4 \alpha_5$, $\xi R \alpha_2 \alpha_3 \eta_1 \eta_2$, $\xi \neq \alpha_1, \alpha_4, \alpha_5$, $\eta_1 \neq \alpha_1, \alpha_4, \alpha_5$, $\eta_2 \neq \alpha_1, \alpha_4, \alpha_5$ imply $\xi R \alpha_2 \alpha_3 \alpha_4 \alpha_5$ or $\xi R \alpha_2 \alpha_3 \alpha_5 \alpha_4$ or $\xi R \alpha_2 \alpha_3 \alpha_5 \alpha_1$ or $\xi R \alpha_2 \alpha_3 \alpha_1 \alpha_5$ or $\xi R \alpha_2 \alpha_3 \alpha_1 \alpha_4$ or $\xi R \alpha_2 \alpha_3 \alpha_4 \alpha_1$.

11. $\alpha_1 R \alpha_2 \alpha_3 \alpha_4 \alpha_5$, $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_4 \neq \alpha_5 \neq \alpha_1$ imply the existence of ε such that $\varepsilon R \alpha_1 \alpha_3 \alpha_4 \alpha_5$, $\varepsilon R \alpha_2 \alpha_1 \alpha_4 \alpha_5$, $\varepsilon R \alpha_2 \alpha_3 \alpha_1 \alpha_5$, $\varepsilon R \alpha_2 \alpha_3 \alpha_4 \alpha_1$.

12. $\alpha_1 R \alpha_2 \alpha_2 \alpha_4 \alpha_5$, $\varepsilon R \alpha_2 \alpha_3 \alpha_4 \alpha_5$, $\alpha_1 R \varepsilon \alpha_3 \alpha_4 \alpha_5$, $\alpha_1 R \alpha_2 \varepsilon \alpha_4 \alpha_5$, $\alpha_1 R \alpha_2 \alpha_3 \varepsilon \alpha_5$, $\alpha_1 R \alpha_2 \alpha_3 \alpha_4 \varepsilon$, $\varepsilon \neq \alpha_1$, α_2, α_3, α_4, α_5 imply: the existence of ξ such that $\xi \bar{R} \varepsilon \alpha_3 \alpha_4 \alpha_5$, $\xi \bar{R} \varepsilon \alpha_3 \alpha_5 \alpha_4$, and the existence of δ', δ'', δ''' such that $\alpha_1 R \alpha_2 \delta' \delta'' \delta'''$ imply $\xi R \alpha_2 \delta' \delta'' \delta'''$, $\alpha_1 R \xi \delta' \delta'' \delta'''$.

II. *Definitions.*

1. ξ is on $\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5$ means, $\alpha_1 R \alpha_2 \alpha_3 \alpha_4 \alpha_5$ and ($\xi R \alpha_2 \alpha_3 \alpha_4 \alpha_5$ or $\alpha_1 R \xi \alpha_3 \alpha_4 \alpha_5$ or $\alpha_1 R \alpha_2 \xi \alpha_4 \alpha_5$ or $\alpha_1 R \alpha_2 \alpha_3 \xi \alpha_5$ or $\alpha_1 R \alpha_2 \alpha_3 \alpha_4 \xi$).

2. ξ is on $\alpha_1 \alpha_2 \alpha_3 \alpha_4$ means, There exists α_5' such that $\alpha_5' R \alpha_1 \alpha_2 \alpha_3 \alpha_4$, and the existence of α_5 such that $\alpha_5 R \alpha_1 \alpha_2 \alpha_3 \alpha_4$ implies $\alpha_5 R \xi \alpha_2 \alpha_3 \alpha_4$ or $\alpha_5 R \alpha_1 \xi \alpha_3 \alpha_4$ or $\alpha_5 R \alpha_1 \alpha_2 \xi \alpha_4$ or $\alpha_5 R \alpha_1 \alpha_2 \alpha_3 \xi$.

3. ξ is on $\alpha_1 \alpha_2 \alpha_3$ means, There exist α_4', α_5' such that $\alpha_4' R \alpha_5' \alpha_1 \alpha_2 \alpha_3$, and the existence of α_4, α_5 such that $\alpha_4 R \alpha_5 \alpha_1 \alpha_2 \alpha_3$ imply that $\alpha_4 R \alpha_5 \xi \alpha_2 \alpha_3$ or $\alpha_4 R \alpha_5 \alpha_1 \xi \alpha_3$ or $\alpha_4 R \alpha_5 \alpha_1 \alpha_2 \xi$.

4. ξ is on $\alpha_1 \alpha_2$ means, $\alpha_1 \neq \alpha_2$ and the existence of α_3, α_4, α_5 such that $\alpha_3 R \alpha_4 \alpha_5 \alpha_1 \alpha_2$ imply that $\alpha_3 R \alpha_4 \alpha_5 \xi \alpha_2$ or $\alpha_3 R \alpha_4 \alpha_5 \alpha_1 \xi$.

Definitions for ξ in the interior of $\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5$, $\alpha_1 \alpha_2 \alpha_3 \alpha_4$, $\alpha_1 \alpha_2 \alpha_3$, $\alpha_1 \alpha_2$ are obtained from the above by substituting "and" for "or". It is readily seen that if ξ is on $\alpha_1 \alpha_2$, ξ is on $\alpha_2 \alpha_1$; if ξ is on $\alpha_1 \alpha_2 \alpha_3$, ξ is on $\alpha_2 \alpha_1 \alpha_3$; but if ξ is on $\alpha_1 \alpha_2 \alpha_3 \alpha_4$, ξ is not necessarily on $\alpha_2 \alpha_1 \alpha_3 \alpha_4$.

Analogous to the definitions under ${}^3R_3$, we may give definitions equivalent to 2, 3, 4. Also we may define:

5. ξ is in the 4-space of the points α_1, α_2, α_3, α_4, α_5 means, ξ is on $\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5$ or $\alpha_2 \alpha_1 \alpha_3 \alpha_4 \alpha_5$.

6. ξ is in the 3-space of the points α_1, α_2, α_3, α_4 means, ξ is on $\alpha_1 \alpha_2 \alpha_3 \alpha_4$ and $\alpha_2 \alpha_1 \alpha_3 \alpha_4$.

7. ξ is in the 2-space of the points α_1, α_2, α_3 means, ξ is on $\alpha_1 \alpha_2 \alpha_3$.

8. ξ is in the 1-space of the points α_1, α_2 means, ξ is on $\alpha_1 \alpha_2$.

Definitions for ξ in the interior of the points α_1, α_2, α_3, α_4, α_5, etc., are readily obtained.

System ${}^nR_n$ $(n \gtreqless 3)$.

A system of axioms for n-dimensional descriptive geometry ($n \gtreqless 3$) is easily constructed on the basis of the foregoing systems. The n-dimensional analogues of axioms 1, 2, 3 of system ${}^4R_4$ are immediately evident. Corresponding to axioms 4, 5, 6 of the same system are a group of $n-1$ axioms which are easily given by means of the rows of the following matrix:

$$\begin{array}{llllllll}
1 & 2 & 3 & 4 & 5\ldots\ldots n-1 & n & n+1 \\
2 & 3 & 1 & 4 & 5\ldots\ldots n-1 & n & n+1 \\
3 & 4 & 1 & 2 & 5\ldots\ldots n-1 & n & n+1 \\
4 & 5 & 1 & 2 & 3\ldots\ldots n-1 & n & n+1 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
n-2 & n-1 & 1 & 2 & 3\ldots\ldots n-3 & n & n+1 \\
n-1 & n & 1 & 2 & 3\ldots\ldots n-3 & n-2 & n+1 \\
n & n+1 & 1 & 2 & 3\ldots\ldots n-3 & n-2 & n-1
\end{array}$$

Thus, from the first and second rows we get,

$$\alpha_1 R \alpha_2 \alpha_3 \ldots\ldots \alpha_{n+1} \text{ implies } \alpha_2 R \alpha_3 \alpha_1 \ldots\ldots \alpha_{n+1};$$

and from the first and third rows we obtain,

$$\alpha_1 R \alpha_2 \alpha_3 \ldots\ldots \alpha_{n+1} \text{ implies } \alpha_3 R \alpha_4 \alpha_1 \alpha_2 \ldots\ldots \alpha_{n+1};$$

and so on. It is clear that the preceding matrix gives us at once a set of generating substitutions for the alternating group on $n+1$ symbols.*

The n-dimensional extensions of axioms 7 and 8 have no difficulty. Corresponding to axioms 9 and 10 is a set of $n-2$ axioms. These are as follows:

1) $\alpha_1 R \alpha_2 \alpha_3 \ldots\ldots \alpha_{n-1} \alpha_n \alpha_{n+1}$, $\xi R \alpha_2 \alpha_3 \ldots\ldots \alpha_{n-1} \alpha_n \eta_1$ imply $\xi R \alpha_2 \alpha_3 \ldots\ldots \alpha_{n-1} \alpha_n \alpha_{n+1}$ or $\xi R \alpha_2 \alpha_3 \ldots\ldots \alpha_{n-1} \alpha_{n+1} \alpha_n$ or $\xi R \alpha_2 \alpha_3 \ldots\ldots \alpha_{n-1} \alpha_n \alpha_1$ or $\xi R \alpha_2 \alpha_3 \ldots\ldots \alpha_{n-1} \alpha_1 \alpha_n$.

2) $\alpha_1 R \alpha_2 \alpha_3 \ldots\ldots \alpha_{n-1} \alpha_n \alpha_{n+1}$, $\xi R \alpha_2 \alpha_3 \ldots\ldots \alpha_{n-1} \eta_1 \eta_2$ imply $\xi R \alpha_2 \alpha_3 \ldots\ldots \alpha_{n-1} \alpha_n \alpha_{n+1}$ or $\xi R \alpha_2 \alpha_3 \ldots\ldots \alpha_{n-1} \alpha_{n+1} \alpha_n$ or $\xi R \alpha_2 \alpha_3 \ldots\ldots \alpha_{n-1} \alpha_{n+1} \alpha_1$ or $\xi R \alpha_2 \alpha_3 \ldots\ldots \alpha_{n-1} \alpha_1 \alpha_{n+1}$ or $\xi R \alpha_2 \alpha_3 \ldots\ldots \alpha_{n-1} \alpha_1 \alpha_n$ or $\xi R \alpha_2 \alpha_3 \ldots\ldots \alpha_{n-1} \alpha_n \alpha_1$.

. . . .

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* Cf. E. H. Moore, *Proc. Lond. Math. Soc.*, XXVIII, p. 357. Our generators do not satisfy Moore's relations directly; they correspond to the standpoint of D. N. Lehmer, *Bulletin Am. Math. Soc.*, XIII, p. 81. It is very easy to find generators which satisfy immediately Moore's relations.

$n-3)$ $\alpha_1 R \alpha_2 \alpha_3 \ldots \alpha_{n-1} \alpha_n \alpha_{n+1}$, $\xi R \alpha_2 \alpha_3 \alpha_4 \eta_1 \eta_2 \ldots \eta_{n-3}$ imply $\xi R \alpha_2 \alpha_3 \alpha_4 \ldots \alpha_{n-1} \alpha_n \alpha_{n+1}$ or $\xi R \alpha_2 \alpha_3 \alpha_4 \ldots \alpha_{n-1} \alpha_{n+1} \alpha_n$ or $\xi R \alpha_2 \alpha_3 \alpha_4 \alpha_6 \ldots \alpha_n \alpha_{n+1} \alpha_1$ or $\xi R \alpha_2 \alpha_3 \alpha_4 \alpha_6 \ldots \alpha_n \alpha_1 \alpha_{n+1}$ or $\xi R \alpha_2 \alpha_3 \alpha_4 \alpha_7 \ldots \alpha_{n+1} \alpha_1 \alpha_5$ or $\xi R \alpha_2 \alpha_3 \alpha_4 \alpha_7 \ldots \alpha_{n+1} \alpha_5 \alpha_1$ or .... or $\xi R \alpha_2 \alpha_3 \alpha_4 \alpha_1 \alpha_5 \alpha_6 \ldots \alpha_{n-1} \alpha_n$ or $\xi R \alpha_2 \alpha_3 \alpha_4 \alpha_1 \alpha_5 \alpha_6 \ldots \alpha_n \alpha_{n-1}$.

$n-2)$ $\alpha_1 R \alpha_2 \alpha_3 \ldots \alpha_{n-1} \alpha_n \alpha_{n+1}$, $\xi R \alpha_2 \alpha_3 \eta_1 \eta_2 \ldots \eta_{n-2}$ imply $\xi R \alpha_2 \alpha_3 \ldots \alpha_n \alpha_{n+1}$ or $\xi R \alpha_2 \alpha_3 \ldots \alpha_{n+1} \alpha_n$ or $\xi R \alpha_2 \alpha_3 \alpha_5 \alpha_6 \ldots \alpha_{n+1} \alpha_1$ or $\xi R \alpha_2 \alpha_3 \alpha_5 \alpha_6 \ldots \alpha_1 \alpha_{n+1}$ or $\xi R \alpha_2 \alpha_3 \alpha_6 \alpha_7 \ldots \alpha_1 \alpha_4$ or $\xi R \alpha_2 \alpha_3 \alpha_6 \alpha_7 \ldots \alpha_4 \alpha_1$ or .... or $\xi R \alpha_2 \alpha_3 \alpha_1 \alpha_4 \alpha_5 \ldots \alpha_{n-1} \alpha_n$ or $\xi R \alpha_2 \alpha_3 \alpha_1 \alpha_4 \alpha_5 \ldots \alpha_n \alpha_{n-1}$.

The n-dimensional extensions of the remaining axioms are readily obtained. We may also easily give definitions analogous to those under system ${}^4R_4$. The independence of the axioms of system ${}^nR_n$ $(n = 2, 3, \ldots)$, and in particular the associated finite systems, we shall discuss elsewhere.*

CHAPTER V.

The Systems ${}^nK_n$ $(n = 1, 2, 3, \ldots)$.

In a former chapter we have given a set of descriptive systems in terms of an alternating relation R. In this chapter we give for $n = 1, 2, 3$ the corresponding systems in terms of the transitive and symmetrical relation K between two $(n+1)$-ads. Each R-system implies the corresponding K-system. Conversely, on the basis of the K-systems the R-systems are definable.† Thus, in order to define $\alpha\, {}^3R_3 \beta \gamma \delta$ on the basis of system ${}^3K_3$ we need only put $\alpha\, {}^3R_3 \beta \gamma \delta = \alpha \beta \gamma \delta\, {}^3K_3\, \alpha_0 \beta_0 \gamma_0 \delta_0$.‡ Notwithstanding the definitional equivalence of the K- and R-systems, it should be noted that as descriptive systems they are essentially distinct;§ for example, they are extended to higher dimensions in very different ways, as we have already pointed out in our introduction.||

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* Cf. the abstract of the author, *Bulletin of the Am. Math. Soc.*, March, 1908, p. 265.

† Compare B. Russell, "The Principles of Mathematics," pp. 166, 235. Russell's insistence on the asymmetry of relations seems to us somewhat strained; compare, for instance, § 225, p. 236, of the work cited.

‡ See the system ${}^3K_3$ in the present chapter.

§ A logical distinction is that between the relative and absolute standpoints. Compare also p. 314 of our paper, *Transactions Am. Math. Soc.*, Vol. X (1909).

|| Compare also Chapter IV.

In the systems ${}^nK_n$ $(n = 1, 2, 3, \ldots)$ we may replace the statement $\alpha_1\alpha_2\ldots\alpha_{n+1}K\beta_1\beta_2\ldots\beta_{n+1}$ by the congruence *

$$\alpha_1\alpha_2\ldots\alpha_{n+1} \equiv \frac{\alpha_1\alpha_2\ldots\alpha_{n+1}}{\beta_1\beta_2\ldots\beta_{n+1}}\beta_1\beta_2\ldots\beta_{n+1}$$

without impairing the validity of the systems.

System ${}^1K_1$. I. *Axioms.*

1. There exists α.

2. The existence of α implies the existence of α_0, β_0 such that $\alpha_0\beta_0K$ or $\beta_0\alpha_0K$.

3. $\alpha\beta K$ implies $\beta\alpha K$.
4. $\alpha\beta K$ implies $\alpha\beta\bar{K}\beta\alpha$.
5. $\alpha\beta K$, $\xi \neq \alpha, \beta$ imply $\xi\beta K$ or $\alpha\xi K$.
6. $\alpha\beta K$, $\xi\beta K$, $\xi \neq \alpha$ imply $\alpha\xi K$.
7. $\xi\beta K\alpha\xi$ implies $\xi\beta K\alpha\beta$.
8. $\alpha\beta K\alpha'\beta'$ implies $\alpha\beta K$.
9. $\alpha\beta K\alpha'\beta'$ implies $\alpha'\beta' K$.
10. $\alpha\beta K\xi\eta$, $\xi\eta K\alpha'\beta'$ imply $\alpha\beta K\alpha'\beta'$.
11. $\alpha\beta K$ implies the existence of ξ such that $\alpha\beta K\xi\alpha$.
12. $\alpha\beta K$ implies the existence of ξ such that $\alpha\beta K\xi\beta$ and $\alpha\beta K\alpha\xi$.
13. $\alpha\beta K$ and $\alpha'\beta' K$ imply $\alpha\beta K\alpha'\beta'$ or $\alpha\beta K\beta'\alpha'$.

II. *Definitions.*

1. $\alpha\beta$ and $\alpha'\beta'$ are collinear means, $\alpha\beta K$, $\alpha'\beta' K$, $\alpha\beta K\alpha'\beta'$ or $\alpha\beta K\beta'\alpha'$.

2. ξ is in the 1-space $\alpha\beta$ means, $\alpha \neq \beta$, $\alpha\beta K\xi\beta$ or $\alpha\beta K\alpha\xi$.

3. ξ is in the interior of the segment $\alpha\beta$ means, $\alpha \neq \beta$, $\alpha\beta K\xi\beta$ and $\alpha\beta K\alpha\xi$. †

System ${}^2K_2$. I. *Axioms.*

1. There exists α.

2. The existence of α implies the existence of α_0, β_0, γ_0 such that $\alpha_0\beta_0\gamma_0 K$ or $\beta_0\alpha_0\gamma_0 K$.

3. $\alpha\beta\gamma K$ implies $\beta\alpha\gamma K$.
4. $\alpha\beta\gamma K$ implies $\alpha\beta\gamma\bar{K}\beta\alpha\gamma$.

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* Compare Grassmann, *Gesammelte Werke*, I, 1, p. 127. The terms of this congruence are symmetric products; see Chapter VI for further details.

† An extension of our system ${}^1K_1$ to two dimensions will be found in *Transactions Am. Math. Soc.*, Vol. X, p. 309.

5. $\alpha\beta\gamma K$ implies $\alpha\beta\gamma K\beta\gamma\alpha$.

6. $\alpha\beta\gamma K$, $\xi \neq \alpha, \beta, \gamma$ imply $\xi\beta\gamma K$ or $\alpha\xi\gamma K$ or $\alpha\beta\xi K$.

7. $\alpha\beta\gamma K$, $\xi\beta\gamma K$, $\xi \neq \alpha$ imply $\alpha\xi\gamma K$ or $\alpha\beta\xi K$.

8. $\xi\beta\gamma K\alpha\xi\gamma$, $\xi\beta\gamma K\alpha\beta\xi$ imply $\xi\beta\gamma K\alpha\beta\gamma$.

8'. $\xi\beta\gamma K\alpha\xi\gamma$, $\alpha\beta\xi\overline{K}$ imply $\xi\beta\gamma K\alpha\beta\gamma$.

9. $\alpha\beta\gamma K\alpha'\beta'\gamma'$ implies $\alpha\beta\gamma K$.

10. $\alpha\beta\gamma K\alpha'\beta'\gamma'$ implies $\alpha'\beta'\gamma' K$.

11. $\alpha\beta\gamma K\xi\eta\zeta$, $\xi\eta\zeta K\alpha'\beta'\gamma'$ imply $\alpha\beta\gamma K\alpha'\beta'\gamma'$.

12. $\alpha\beta\gamma K$ implies the existence of ξ such that $\alpha\beta\gamma K\xi\alpha\gamma$ and $\alpha\beta\gamma K\xi\beta\alpha$.

13. $\alpha\beta\gamma K$, $\xi\beta\gamma K\alpha\xi\gamma$, $\xi\beta\gamma K\alpha\beta\xi$, $\xi\beta\gamma K\alpha\beta\gamma$ imply: the existence of η such that $\eta\xi\gamma\overline{K}$, and the existence of γ' such that $\alpha\beta\gamma' K$ implies $\alpha\beta\gamma' K\eta\beta\gamma'$ and $\alpha\beta\gamma' K\alpha\eta\gamma'$.

14. $\alpha\beta\gamma K$, $\alpha'\beta'\gamma' K$ imply $\alpha\beta\gamma K\alpha'\beta'\gamma'$ or $\alpha\beta\gamma K\beta'\alpha'\gamma'$.

II. *Definitions.*

1. $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ are coplanar means, $\alpha\beta\gamma K$, $\alpha'\beta'\gamma' K$, $\alpha\beta\gamma K\alpha'\beta'\gamma'$ or $\alpha\beta\gamma K\beta'\alpha'\gamma'$.

2. ξ is in the 2-space $\alpha\beta\gamma$ means, $\alpha\beta\gamma K$, $\alpha\beta\gamma K\xi\beta\gamma$ or $\alpha\beta\gamma K\alpha\xi\gamma$ or $\alpha\beta\gamma K\alpha\beta\xi$.

3. ξ is in the 1-space $\alpha\beta$ means, $\alpha \neq \beta$ and $\alpha\beta\xi\overline{K}$.

4. ξ is in the interior of the triangle $\alpha\beta\gamma$ means, $\alpha\beta\gamma K$, $\alpha\beta\gamma K\xi\beta\gamma$, $\alpha\beta\gamma K\alpha\xi\gamma$, $\alpha\beta\gamma K\alpha\beta\xi$.

5. ξ is in the interior of the segment $\alpha\beta$ means, $\alpha \neq \beta$ and the existence of γ' such that $\alpha\beta\gamma' K$ implies $\alpha\beta\gamma' K\xi\beta\gamma'$, $\alpha\beta\gamma' K\alpha\xi\gamma'$.

System $^3K_3$. I. *Axioms.*

1. There exists α.

2. The existence of α implies the existence of α_0, β_0, γ_0, δ_0 such that $\alpha_0\beta_0\gamma_0\delta_0 K$ or $\beta_0\alpha_0\gamma_0\delta_0 K$.

3. $\alpha\beta\gamma\delta K$ implies $\beta\alpha\gamma\delta K$.

4. $\alpha\beta\gamma\delta K$ implies $\alpha\beta\gamma\delta\overline{K}\beta\alpha\gamma\delta$.

5. $\alpha\beta\gamma\delta K$ implies $\alpha\beta\gamma\delta K\beta\gamma\alpha\delta$.

6. $\alpha\beta\gamma\delta K$ implies $\alpha\beta\gamma\delta K\gamma\delta\alpha\beta$.

7. $\alpha\beta\gamma\delta K$, $\xi \neq \alpha, \beta, \gamma, \delta$ imply $\xi\beta\gamma\delta K$ or $\alpha\xi\gamma\delta K$ or $\alpha\beta\xi\delta K$ or $\alpha\beta\gamma\xi K$.

8. $\alpha\beta\gamma\delta K$, $\xi\beta\gamma\delta K$, $\xi \neq \alpha$ imply $\alpha\xi\gamma\delta K$ or $\alpha\beta\xi\delta K$ or $\alpha\beta\gamma\xi K$.

9. $\alpha\beta\gamma\delta K$, $\xi\beta\gamma\eta K$, $\xi \neq \alpha, \delta$, $\eta \neq \alpha, \delta$ imply $\xi\beta\gamma\delta K$ or $\xi\beta\gamma\alpha K$.

10. $\xi\beta\gamma\delta K\alpha\xi\gamma\delta$, $\xi\beta\gamma\delta K\alpha\beta\xi\delta$, $\xi\beta\gamma\delta K\alpha\beta\gamma\xi$ imply $\xi\beta\gamma\delta K\alpha\beta\gamma\delta$.

10′. $\xi\beta\gamma\delta K\alpha\xi\gamma\delta$, $\xi\beta\gamma\delta K\alpha\beta\xi\delta$, $\alpha\beta\gamma\xi\overline{K}$ imply $\xi\beta\gamma\delta K\alpha\beta\gamma\delta$.

10″. $\xi\beta\gamma\delta K\alpha\xi\gamma\delta$, $\alpha\beta\xi\delta\overline{K}$, $\alpha\beta\gamma\xi\overline{K}$ imply $\xi\beta\gamma\delta K\alpha\beta\gamma\delta$.

11. $\alpha\beta\gamma\delta K\alpha'\beta'\gamma'\delta'$ implies $\alpha\beta\gamma\delta K$.

12. $\alpha\beta\gamma\delta K\alpha'\beta'\gamma'\delta'$ implies $\alpha'\beta'\gamma'\delta' K$.

13. $\alpha\beta\gamma\delta K\xi\eta\zeta\tau$, $\xi\eta\zeta\tau K\alpha'\beta'\gamma'\delta'$ imply $\alpha\beta\gamma\delta K\alpha'\beta'\gamma'\delta'$.

14. $\alpha\beta\gamma\delta K$ implies the existence of ξ such that $\alpha\beta\gamma\delta K\xi\alpha\gamma\delta$, $\alpha\beta\gamma\delta K\xi\beta\alpha\delta$, $\alpha\beta\gamma\delta K\xi\beta\gamma\alpha$.

15. $\alpha\beta\gamma\delta K$, $\xi\beta\gamma\delta K\alpha\xi\gamma\delta$, $\xi\beta\gamma\delta K\alpha\beta\xi\delta$, $\xi\beta\gamma\delta K\alpha\beta\gamma\xi$, $\xi\beta\gamma\delta K\alpha\beta\gamma\delta$ imply: the existence of η such that $\eta\gamma\xi\delta K$, and the existence of γ', δ' such that $\alpha\beta\gamma'\delta' K$ implies $\alpha\beta\gamma'\delta' K\alpha\eta\gamma'\delta'$ and $\alpha\beta\gamma'\delta' K\eta\beta\gamma'\delta'$.

16. $\alpha\beta\gamma\delta K$, $\alpha'\beta'\gamma'\delta' K$ imply $\alpha\beta\gamma\delta K\alpha'\beta'\gamma'\delta'$ or $\alpha\beta\gamma\delta K\beta'\alpha'\gamma'\delta'$.

II. *Definitions.*

1. $\alpha\beta\gamma\delta$, $\alpha'\beta'\gamma'\delta'$ are cospatial means, $\alpha\beta\gamma\delta K$, $\alpha'\beta'\gamma'\delta' K$, $\alpha\beta\gamma\delta K\alpha'\beta'\gamma'\delta'$ or $\alpha\beta\gamma\delta K\beta'\alpha'\gamma'\delta'$.

2. ξ is in the 3-space $\alpha\beta\gamma\delta$ means, $\alpha\beta\gamma\delta K$, $\alpha\beta\gamma\delta K\xi\beta\gamma\delta$ or $\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$ or $\alpha\beta\gamma\delta K\alpha\beta\xi\delta$ or $\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$.

3. ξ is in the 2-space $\alpha\beta\gamma$ means, $\alpha\beta\gamma\xi\overline{K}$ and there exists a δ such that $\alpha\beta\gamma\delta K$.

4. ξ is in the 1-space $\alpha\beta$ means, $\alpha \neq \beta$ and the existence of δ' implies $\alpha\beta\xi\delta'\overline{K}$.

5. ξ is in the interior of the tetrahedron $\alpha\beta\gamma\delta$ means, $\alpha\beta\gamma\delta K$, $\alpha\beta\gamma\delta K\xi\beta\gamma\delta$, $\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$, $\alpha\beta\gamma\delta K\alpha\beta\xi\delta$, $\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$; that is, $\alpha\beta\gamma\delta K\xi\beta\gamma\delta K\alpha\xi\gamma\delta K\alpha\beta\xi\delta K\alpha\beta\gamma\xi$, where the order of the terms of the latter statement is, under the axioms, immaterial.

6. ξ is in the interior of the triangle $\alpha\beta\gamma$ means, There exists δ_1 such that $\alpha\beta\gamma\delta_1 K$; and the existence of δ' such that $\alpha\beta\gamma\delta' K$ implies $\alpha\beta\gamma\delta' K\xi\beta\gamma\delta'$, $\alpha\beta\gamma\delta' K\alpha\xi\gamma\delta'$, $\alpha\beta\gamma\delta' K\alpha\beta\xi\delta'$, $\alpha\beta\gamma\delta' K\alpha\beta\gamma\xi$; that is, $\alpha\beta\gamma\delta' K\xi\beta\gamma\delta' K\alpha\xi\gamma\delta' K\alpha\beta\xi\delta'$.

7. ξ is in the interior of the segment $\alpha\beta$ means, $\alpha \neq \beta$ and the existence of γ', δ' such that $\alpha\beta\gamma'\delta' K$ implies $\alpha\beta\gamma'\delta' K\xi\beta\gamma'\delta'$, $\alpha\beta\gamma'\delta' K\alpha\xi\gamma'\delta'$; that is, $\alpha\beta\gamma'\delta' K\xi\beta\gamma'\delta' K\alpha\xi\gamma'\delta'$.

III. *Theorems.*

On the basis of the system ${}^3R_3$* we may define: $\alpha\beta\gamma\delta K$ means, $\alpha R\beta\gamma\delta$ or $\beta R\alpha\gamma\delta$; $\alpha\beta\gamma\delta K\alpha'\beta'\gamma'\delta'$ means, ($\alpha R\beta\gamma\delta$ and $\alpha' R\beta'\gamma'\delta'$) or ($\beta R\alpha\gamma\delta$ and $\beta' R\alpha'\gamma'\delta'$). Then it can be easily verified that the validity of system ${}^3R_3$ implies that of the system ${}^3K_3$. Conversely, on the basis of the system ${}^3K_3$ we may define: $\alpha R\beta\gamma\delta$ means, $\alpha\beta\gamma\delta K\alpha_0\beta_0\gamma_0\delta_0$. Accordingly we term the relations ${}^3R_3$ and ${}^3K_3$, "absolute" and "relative" right-handedness respectively.

By means of the system ${}^3K_3$ the usual theorems of three-dimensional descriptive geometry can be established, excluding, of course, the property of continuity. This we proceed to show in detail.

Theorem 1. If $\alpha\beta\gamma\delta K$, then $\alpha, \beta, \gamma, \delta$ are distinct.

By axiom 4, $\alpha\beta\gamma\delta K$ implies $\alpha\beta\gamma\delta\overline{K}\beta\alpha\gamma\delta$. Hence $\alpha\neq\beta$. By axiom 5, $\alpha\beta\gamma\delta K$ implies $\alpha\beta\gamma\delta K\beta\gamma\alpha\delta$. Hence by axiom 12, $\beta\gamma\alpha\delta K$; *i. e.*, by axiom 4, $\beta\gamma\alpha\delta\overline{K}\gamma\beta\alpha\delta$ and hence $\beta\neq\gamma$, and so on.

Theorem 2. There exist four distinct points.

By axioms 1 and 2, and theorem 1.

Theorem 3. If $\alpha\beta\gamma\delta K\lambda\mu\nu\omega$, then $\lambda\mu\nu\omega K\alpha\beta\gamma\delta$; *i. e.*, the relation K is symmetrical.

Since $\alpha\beta\gamma\delta K\lambda\mu\nu\omega$, by axioms 11 and 12, $\alpha\beta\gamma\delta K$, $\lambda\mu\nu\omega K$. Hence by axiom 16, $\lambda\mu\nu\omega K\alpha\beta\gamma\delta$ or $\lambda\mu\nu\omega K\beta\alpha\gamma\delta$. If the latter, then by axiom 13 and the hypothesis, $\alpha\beta\gamma\delta K\beta\alpha\gamma\delta$. But this is impossible by axiom 4.

Theorem 4. If $\alpha\beta\gamma\delta K\lambda\mu\nu\omega$, then $\alpha\beta\gamma\delta\overline{K}\mu\lambda\nu\omega$, $\beta\gamma\alpha\delta K\mu\nu\lambda\omega$, $\gamma\delta\alpha\beta K\nu\omega\lambda\mu$.

If $\alpha\beta\gamma\delta K\mu\lambda\nu\omega$, then since the K is symmetrical and transitive by theorem 3 and axiom 13, $\lambda\mu\nu\omega K\mu\lambda\nu\omega$, which contradicts axiom 4.

Since $\alpha\beta\gamma\delta K\lambda\mu\nu\omega$, by axiom 12, $\lambda\mu\nu\omega K$. Hence by axiom 5, $\lambda\mu\nu\omega K\mu\nu\lambda\omega$. Therefore $\alpha\beta\gamma\delta K\mu\nu\lambda\omega$. Again, by axiom 11, $\alpha\beta\gamma\delta K$; hence by axiom 5, $\alpha\beta\gamma\delta K\beta\gamma\alpha\delta$. Since $\alpha\beta\gamma\delta K\mu\nu\lambda\omega$ and $\alpha\beta\gamma\delta K\beta\gamma\alpha\delta$, $\beta\gamma\alpha\delta K\mu\nu\lambda\omega$.

The remaining part of theorem 4 is proved in an analogous manner.

Theorem 4'. If $\alpha\beta\gamma\delta K$, then the order of the points $\alpha, \beta, \gamma, \delta$ is immaterial.

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* Compare Chapter IV.

Proof follows at once from axioms 3, 5, 6 through the application of axiom 12.

Theorem 5. If $\alpha\beta\gamma\delta K$, $\xi \neq \eta$, then $\xi\eta\alpha\beta K$ or $\xi\eta\beta\gamma K$ or $\xi\eta\gamma\delta K$ or $\xi\eta\alpha\gamma K$ or $\xi\eta\beta\delta K$ or $\xi\eta\alpha\delta K$.

Since $\alpha\beta\gamma\delta K$, if $\xi \neq \alpha, \beta, \gamma, \delta$, we have by axiom 7, $\xi\beta\gamma\delta K$ or $\alpha\xi\gamma\delta K$ or $\alpha\beta\xi\delta K$ or $\alpha\beta\gamma\xi K$. Suppose $\xi\beta\gamma\delta K$; then $\eta\beta\gamma\delta K$ or $\xi\eta\gamma\delta K$ or $\xi\beta\eta\delta K$ or $\xi\beta\gamma\eta K$. If $\xi\beta\gamma\delta K$ and $\eta\beta\gamma\delta K$, then by axiom 8 and theorem 4′ theorem follows. Similarly for the other cases.

Theorem 6. If $\alpha\beta\gamma\delta K$, $\xi\beta\gamma\delta\overline{K}$, $\xi\beta\gamma\alpha\overline{K}$, then for any Δ,* $\xi\beta\gamma\Delta\overline{K}$.

For suppose that $\xi\beta\gamma\delta_1 K$. Then theorem follows by the application of axiom 9 and theorem 4′.

Theorem 7. If $\alpha\beta\xi\delta_1 K$, $\alpha\beta\xi\gamma\overline{K}$, $\alpha\beta\xi\delta\overline{K}$, then $\alpha\beta\gamma\delta\overline{K}$.

For if $\alpha\beta\gamma\delta K$, since $\alpha\beta\xi\gamma\overline{K}$, $\alpha\beta\xi\delta\overline{K}$, by theorem 6, $\alpha\beta\xi\Delta\overline{K}$ for any Δ, which contradicts the hypothesis.

For brevity, we introduce the following definitions: †

1. $\alpha\beta\gamma\delta K_2$ means, $\alpha\beta\gamma\delta\overline{K}$, but there exists a δ_1 such that $\alpha\beta\gamma\delta_1 K$.
2. $\alpha\beta\gamma\delta K_1$ means, $\alpha \neq \beta$ and there is no δ_1 such that $\alpha\beta\gamma\delta_1 K$.

Thus, theorem 7 says, $\alpha\beta\xi\gamma K_2$, $\alpha\beta\xi\delta K_2$ imply $\alpha\beta\gamma\delta\overline{K}$.

Theorem 8. If $\alpha\beta\gamma\delta K$, $\alpha\beta\xi\Delta K_1$, $\xi \neq \alpha$, then $\xi\alpha\gamma\delta K$.

Since $\alpha\beta\gamma\delta K$, if $\xi \neq \alpha, \beta, \gamma, \delta$, by axiom 7, $\xi\beta\gamma\delta K$ or $\alpha\xi\gamma\delta K$ or $\alpha\beta\xi\delta K$ or $\alpha\beta\gamma\xi K$. Hence $\xi\beta\gamma\delta K$ or $\alpha\xi\gamma\delta K$. If $\xi\beta\gamma\delta K$, since $\alpha\beta\gamma\delta K$, by axiom 8, $\alpha\xi\gamma\delta K$ or $\alpha\beta\xi\delta K$ or $\alpha\beta\gamma\xi K$. Hence $\alpha\xi\gamma\delta K$; *i. e.*, by axiom 3, $\xi\alpha\gamma\delta K$.

Theorem 9. If $\alpha\beta\gamma\delta K_2$, $\alpha\delta\xi\Delta K$, $\xi \neq \delta$, then $\alpha\beta\gamma\xi K_2$.

Suppose $\alpha\beta\gamma\xi K$. Then if $\delta \neq \alpha, \beta, \gamma, \xi$ we have, by axiom 7, $\delta\beta\gamma\xi K$ or $\alpha\beta\delta\xi K$ or $\alpha\delta\gamma\xi K$ or $\alpha\beta\gamma\delta K$. Since $\alpha\delta\xi\gamma\overline{K}$ and $\alpha\delta\xi\beta\overline{K}$, $\alpha\delta\gamma\xi\overline{K}$ and $\alpha\beta\delta\xi\overline{K}$; also by hypothesis, $\alpha\beta\gamma\delta\overline{K}$. Hence $\delta\beta\gamma\xi K$; *i. e.*, $\delta\xi\beta\gamma K$. Hence, by theorem 8, since $\delta\xi\alpha\Delta K_1$, $\alpha\delta\beta\gamma K$, which contradicts the hypothesis. Hence $\alpha\beta\gamma\xi\overline{K}$; *i. e.*, since $\alpha\beta\gamma\delta K_2$, $\alpha\beta\gamma\xi K_2$.

Theorem 10. If $\alpha\beta\gamma\xi K_2$, $\alpha\beta\gamma\eta K_2$, and $\xi\eta\zeta\Delta K_1$, then $\alpha\beta\gamma\zeta K_2$.

Suppose $\alpha\beta\gamma\zeta K$. Then since $\xi \neq \eta$, we have, by theorem 5, $\xi\eta\zeta\alpha K$ or $\xi\eta\zeta\beta K$ or $\xi\eta\zeta\gamma K$ or $\xi\eta\alpha\beta K$ or $\xi\eta\alpha\gamma K$ or $\xi\eta\beta\gamma K$. Now by hypothesis, $\xi\eta\zeta\alpha\overline{K}$, $\xi\eta\zeta\beta\overline{K}$, $\xi\eta\zeta\gamma\overline{K}$. Also, since $\alpha\beta\gamma\eta K_2$ and $\alpha\beta\gamma\xi K_2$,

* An arbitrary point will sometimes be denoted by Δ.

† We remind the reader, in this connection, that $\alpha\beta\gamma\delta K$ was defined (Chap. III) to be, $\alpha\beta\gamma\delta K\alpha\beta\gamma\delta$.

we have, by theorem 7, $\alpha\beta\xi\eta\overline{K}$, $\alpha\gamma\xi\eta\overline{K}$, $\beta\gamma\xi\eta\overline{K}$. Hence $\alpha\beta\gamma\zeta\overline{K}$; *i. e*,. since $\alpha\beta\gamma\xi K_2$, $\alpha\beta\gamma\zeta K_2$.

Theorem 11. If $\alpha\beta\xi\Delta K_1$ and $\alpha\beta\eta\Delta K_1$ and $\xi\neq\eta$, then $\xi\eta\alpha\Delta K_1$ and $\xi\eta\beta\Delta K_1$.

We may suppose $\xi\neq\alpha,\beta$ and $\eta\neq\alpha,\beta$. Since $\alpha\neq\beta$, there exist, by axioms 1, 2 and theorem 5, two points γ,δ such that $\alpha\beta\gamma\delta K$. Then $\alpha\beta\gamma\xi K_2$, $\alpha\beta\delta\xi K_2$, $\alpha\beta\gamma\eta K_2$, $\alpha\beta\delta\eta K_2$. Since $\alpha\beta\gamma\delta K$, $\xi\neq\beta$, we have, by theorem 5, $\xi\beta\gamma\delta K$ or $\xi\beta\delta\alpha K$ or $\xi\beta\alpha\gamma K$. Hence, $\xi\beta\gamma\delta K$. Similarly, $\eta\beta\gamma\delta K$. From $\alpha\beta\gamma\xi K_2$ and $\alpha\beta\gamma\eta K_2$ we have $\xi\beta\gamma\eta\overline{K}$; *i. e.*, since $\xi\beta\gamma\delta K$, $\xi\beta\gamma\eta K_2$. Similarly, $\xi\beta\delta\eta K_2$. Hence, since $\xi\beta\gamma\delta K$, $\xi\beta\gamma\eta K_2$, $\xi\beta\delta\eta K_2$, we have, by theorem 6, $\xi\eta\beta\Delta K_1$. In a similar manner we show that $\xi\eta\alpha\Delta K_1$.

Theorem 12. If $\alpha\beta\gamma\xi K_2$, $\alpha\beta\gamma\eta K_2$, $\alpha\beta\gamma\zeta K_2$ and $\xi\eta\zeta\zeta K_2$, then $\xi\eta\zeta\alpha K_2$, $\xi\eta\zeta\beta K_2$, $\xi\eta\zeta\gamma K_2$.

It will suffice to prove that $\xi\eta\zeta\alpha K_2$. If $\xi\eta\zeta\alpha\overline{K}$, then $\xi\eta\zeta\alpha K_2$. Suppose now $\xi\eta\zeta\alpha K$. Then from $\alpha\beta\gamma\eta K_2$ and $\alpha\beta\gamma\zeta K_2$, we have, by theorem 7, $\alpha\beta\gamma\zeta\overline{K}$. But $\alpha\eta\zeta\xi K$. Hence $\alpha\eta\zeta\beta K_2$. Similarly, $\alpha\xi\zeta\beta K_2$. Therefore, by theorem 6, $\alpha\beta\zeta\Delta K_1$. Similarly, $\alpha\gamma\zeta\Delta K_1$. Since $\zeta\neq\alpha$, we have then, by theorem 11, $\alpha\beta\gamma\Delta K_1$. Since this contradicts the hypothesis, we must have $\xi\eta\zeta\alpha\overline{K}$; *i. e.*, $\xi\eta\zeta\alpha K_2$.

Theorem 13. If $\alpha\beta\gamma\delta K$ and ξ is any point, then $\alpha\beta\gamma\delta K\xi\beta\gamma\delta$ or $\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$ or $\alpha\beta\gamma\delta K\alpha\beta\xi\delta$ or $\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$.

We suppose, first, that $\xi\beta\gamma\delta K$, $\alpha\xi\gamma\delta K$, $\alpha\beta\xi\delta K$, $\alpha\beta\gamma\xi K$. Then if the theorem is not verified, by the application of axiom 16, we must have $\alpha\beta\gamma\delta K\xi\alpha\gamma\delta$, $\alpha\beta\gamma\delta K\beta\xi\gamma\delta$, $\alpha\beta\gamma\delta K\beta\alpha\xi\delta$, $\alpha\beta\gamma\delta K\beta\alpha\gamma\xi$. Since the K is symmetrical and transitive, we have $\xi\alpha\gamma\delta K\beta\xi\gamma\delta$, $\xi\alpha\gamma\delta K\beta\alpha\xi\delta$, $\xi\alpha\gamma\delta K\beta\alpha\gamma\xi$. Hence, by axiom 10, $\xi\alpha\gamma\delta K\beta\alpha\gamma\delta$. Since $\alpha\beta\gamma\delta K\xi\alpha\gamma\delta$, we have $\alpha\beta\gamma\delta K\beta\alpha\gamma\delta$, which contradicts axiom 4.

If $\xi\beta\gamma\delta K$, $\alpha\xi\gamma\delta K$, $\alpha\beta\xi\delta K$, $\alpha\beta\gamma\xi\overline{K}$, then we proceed as above, but employ axiom 10′ instead of axiom 10. Similarly, the theorem is verified by the use of axiom 10″ if $\xi\beta\gamma\delta K$, $\alpha\xi\gamma\delta K$, $\alpha\beta\xi\delta\overline{K}$, $\alpha\beta\gamma\xi\overline{K}$.

Finally, if $\xi\beta\gamma\delta K$, $\alpha\xi\gamma\delta\overline{K}$, $\alpha\beta\xi\delta\overline{K}$, $\alpha\beta\gamma\xi\overline{K}$, we have $\xi=\alpha$ by axiom 8.

Theorem 14. If $\alpha\beta\gamma\delta K$, $\xi\beta\gamma\delta K$, $\alpha\xi\gamma\delta K$, $\alpha\beta\xi\delta K$, $\alpha\beta\gamma\xi K$, then ξ is in the interior of one of precisely fifteen compartments associated with $\alpha\beta\gamma\delta$.

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By theorem 13, we have, since $\alpha\beta\gamma\delta K$, $\alpha\beta\gamma\delta K\xi\beta\gamma\delta$ or $\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$ or $\alpha\beta\gamma\delta K\alpha\beta\xi\delta$ or $\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$. Suppose $\alpha\beta\gamma\delta K\xi\beta\gamma\delta$. That is, by axiom 12, $\xi\beta\gamma\delta K$. Since $\alpha\beta\gamma\delta K$ and $\xi\beta\gamma\delta K$, by axiom 8, $\alpha\xi\gamma\delta K$ or $\alpha\beta\xi\delta K$ or $\alpha\beta\gamma\xi K$. Hence, by axiom 16, ($\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$ or $\alpha\beta\gamma\delta K\xi\alpha\gamma\delta$) or ($\alpha\beta\gamma\delta K\alpha\beta\xi\delta$ or $\alpha\beta\gamma\delta K\beta\alpha\xi\delta$) or ($\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$ or $\alpha\beta\gamma\delta K\beta\alpha\gamma\xi$). We have then the following eight possibilities:

(1)	(2)	(3)	(4)
$\alpha\beta\gamma\delta K\xi\beta\gamma\delta$	$\alpha\beta\gamma\delta K\xi\beta\gamma\delta$	$\alpha\beta\gamma\delta K\xi\beta\gamma\delta$	$\alpha\beta\gamma\delta K\xi\beta\gamma\delta$
$\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$	$\alpha\beta\gamma\delta K\alpha\xi\delta\gamma$	$\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$	$\alpha\beta\gamma\delta K\alpha\beta\xi\delta$
$\alpha\beta\gamma\delta K\alpha\beta\xi\delta$	$\alpha\beta\gamma\delta K\alpha\beta\delta\xi$	$\alpha\beta\gamma\delta K\alpha\beta\delta\xi$	$\alpha\beta\gamma\delta K\alpha\beta\xi\gamma$
$\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$	$\alpha\beta\gamma\delta K\alpha\beta\xi\gamma$	$\alpha\beta\gamma\delta K\alpha\beta\xi\gamma$	$\alpha\beta\gamma\delta K\alpha\xi\delta\gamma$

(5)	(6)	(7)	(8)
$\alpha\beta\gamma\delta K\xi\beta\gamma\delta$	$\alpha\beta\gamma\delta K\xi\beta\gamma\delta$	$\alpha\beta\gamma\delta K\xi\beta\gamma\delta$	$\alpha\beta\gamma\delta K\xi\beta\gamma\delta$
$\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$	$\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$	$\alpha\beta\gamma\delta K\alpha\beta\xi\delta$	$\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$
$\alpha\beta\gamma\delta K\alpha\xi\delta\gamma$	$\alpha\beta\gamma\delta K\alpha\beta\xi\delta$	$\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$	$\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$
$\alpha\beta\gamma\delta K\alpha\beta\delta\xi$	$\alpha\beta\gamma\delta K\alpha\beta\xi\gamma$	$\alpha\beta\gamma\delta K\alpha\xi\delta\gamma$	$\alpha\beta\gamma\delta K\alpha\beta\delta\xi$

If $\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$, we again get eight possibilities, leaving, however, only the following new ones:

(9)	(10)	(11)	(12)
$\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$	$\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$	$\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$	$\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$
$\alpha\beta\gamma\delta K\xi\beta\delta\gamma$	$\alpha\beta\gamma\delta K\alpha\beta\xi\delta$	$\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$	$\alpha\beta\gamma\delta K\alpha\beta\xi\delta$
$\alpha\beta\gamma\delta K\alpha\beta\delta\xi$	$\alpha\beta\gamma\delta K\alpha\beta\xi\gamma$	$\alpha\beta\gamma\delta K\xi\beta\delta\gamma$	$\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$
$\alpha\beta\gamma\delta K\alpha\beta\xi\gamma$	$\alpha\beta\gamma\delta K\xi\beta\delta\gamma$	$\alpha\beta\gamma\delta K\alpha\beta\delta\xi$	$\alpha\beta\gamma\delta K\xi\beta\delta\gamma$

If $\alpha\beta\gamma\delta K\alpha\beta\xi\delta$, from the corresponding eight possibilities, we get two cases:

(13)	(14)
$\alpha\beta\gamma\delta K\alpha\beta\xi\delta$	$\alpha\beta\gamma\delta K\alpha\beta\xi\delta$
$\alpha\beta\gamma\delta K\xi\beta\delta\gamma$	$\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$
$\alpha\beta\gamma\delta K\alpha\xi\delta\gamma$	$\alpha\beta\gamma\delta K\xi\beta\delta\gamma$
$\alpha\beta\gamma\delta K\alpha\beta\xi\gamma$	$\alpha\beta\gamma\delta K\alpha\xi\delta\gamma$

Finally, if $\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$, we get as the only new case:

(15)
$\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$
$\alpha\beta\gamma\delta K\xi\beta\delta\gamma$
$\alpha\beta\gamma\delta K\alpha\xi\delta\gamma$
$\alpha\beta\gamma\delta K\alpha\beta\delta\xi$

The theorem therefore follows. It is plain that of the fifteen compartments indicated above there is a compartment associated with every vertex, every face, and every edge of $\alpha\gamma\beta\delta$, in addition to $\alpha\beta\gamma\delta$ itself. Thus:

1) ξ is in the interior of $\alpha\beta\gamma\delta$,
2) ξ is in the interior of the compartment at α of $\alpha\beta\gamma\delta$, say C_α,
3) ξ is in the interior of the compartment at $\alpha\beta$ of $\alpha\beta\gamma\delta$, $C_{\alpha\beta}$,
4) ξ is in the interior of the compartment at $\alpha\gamma$ of $\alpha\beta\gamma\delta$, $C_{\alpha\gamma}$,
5) ξ is in the interior of the compartment at $\alpha\delta$ of $\alpha\beta\gamma\delta$, $C_{\alpha\delta}$,
6) ξ is in the interior of the compartment at $\alpha\beta\gamma$ of $\alpha\beta\gamma\delta$, $C_{\alpha\beta\gamma}$,
7) ξ is in the interior of the compartment at $\alpha\gamma\delta$ of $\alpha\beta\gamma\delta$, $C_{\alpha\gamma\delta}$,
8) ξ is in the interior of the compartment at $\alpha\beta\delta$ of $\alpha\beta\gamma\delta$, $C_{\alpha\beta\delta}$,

and so on.

Theorem 15. If $\beta \neq \gamma$, there exists a ξ' in the interior of $\beta\gamma$.

By theorem 5 and axiom 2, there exist two points γ, δ such that $\alpha\beta\gamma\delta K$. Hence, by axiom 14, there is a ξ such that $\alpha\beta\gamma\delta K\xi\alpha\gamma\delta$, $\alpha\beta\gamma\delta K\xi\beta\alpha\delta$, $\alpha\beta\gamma\delta K\xi\beta\gamma\alpha$. Hence, by axiom 10, $\alpha\beta\gamma\delta K\xi\beta\gamma\delta$. Therefore, by axiom 15, there is a point η in the interior of $\beta\gamma$ such that $\xi\alpha\delta\eta K_2$.

Theorem 15′. If ξ is in the interior of $\beta\gamma$, ξ is in the interior of $\gamma\beta$ and β is not in the interior of $\xi\gamma$.

Since ξ is in the interior of $\beta\gamma$, by definition 7, for any two points δ_1, δ_2 such that $\beta\gamma\delta_1\delta_2 K$, $\beta\gamma\delta_1\delta_2 K\xi\gamma\delta_1\delta_2 K\beta\xi\delta_1\delta_2$. Hence, by theorem 4, for any two points δ_1, δ_2 such that $\gamma\beta\delta_1\delta_2 K$, $\gamma\beta\delta_1\delta_2 K\gamma\xi\delta_1\delta_2 K\xi\beta\delta_1\delta_2$.

Also β is not in the interior of $\xi\gamma$, for if so, for any two points δ_1', δ_2' such that $\xi\gamma\delta_1'\delta_2' K$, $\xi\gamma\delta_1'\delta_2' K\beta\gamma\delta_1'\delta_2' K\xi\beta\delta_1'\delta_2'$. But $\xi\gamma\delta_1\delta_2 K$, since ξ is in the interior of $\beta\gamma$; hence $\xi\gamma\delta_1\delta_2 K\beta\gamma\delta_1\delta_2 K\xi\beta\delta_1\delta_2$, which is impossible by axioms 4, 11, 12.

Theorem 16. If ξ is in the interior of $\alpha\beta$ and η is in the interior of $\alpha\xi$, then η is in the interior of $\alpha\beta$.

By definition and the hypothesis, we have, for any two points δ_1, δ_2 such that $\alpha\beta\delta_1\delta_2 K$, $\alpha\beta\delta_1\delta_2 K\xi\beta\delta_1\delta_2 K\alpha\xi\delta_1\delta_2$, and for any two points δ_1', δ_2' such that $\alpha\xi\delta_1'\delta_2' K$, $\alpha\xi\delta_1'\delta_2' K\eta\xi\delta_1'\delta_2' K\alpha\eta\delta_1'\delta_2'$. Since $\alpha\xi\delta_1\delta_2 K$, $\alpha\xi\delta_1\delta_2 K\eta\xi\delta_1\delta_2 K\alpha\eta\delta_1\delta_2$. Hence $\eta\xi\delta_1\delta_2 K\xi\beta\delta_1\delta_2$, $\eta\beta\xi\delta_1\overline{K}$, $\eta\beta\xi\delta_2\overline{K}$; therefore, by axiom 10″, $\eta\xi\delta_1\delta_2 K\eta\beta\delta_1\delta_2$. Hence $\alpha\beta\delta_1\delta_2 K\eta\beta\delta_1\delta_2 K\alpha\eta\delta_1\delta_2$, *i. e.*, η is in the interior of $\alpha\beta$.

Theorem 17. If $\alpha\beta\gamma\delta K_2$, ξ is in the interior of $\alpha\beta$ and η is in the interior of $\gamma\xi$, then η is in the interior of $\alpha\beta\gamma$.

Let $\alpha\beta\gamma\delta_1 K$. Then since ξ is in the interior of $\alpha\beta$, we have,

$$\alpha\beta\gamma\delta_1 K \xi\beta\gamma\delta_1 K \alpha\xi\gamma\delta_1. \tag{1}$$

Since $\alpha\xi\gamma\delta_1 K$ and η is in the interior of $\xi\gamma$, we have,

$$\xi\gamma\alpha\delta_1 K \eta\gamma\alpha\delta_1 K \xi\eta\alpha\delta_1. \tag{2}$$

Since $\alpha\xi\eta\delta_1 K$ and $\alpha\beta\xi\Delta K_1$, by theorem 8, $\beta\alpha\eta\delta_1 K$. Therefore, since ξ is in the interior of $\alpha\beta$,

$$\alpha\beta\eta\delta_1 K \xi\beta\eta\delta_1 K \alpha\xi\eta\delta_1. \tag{3}$$

Further, since $\xi\gamma\beta\delta_1 K$ and η is in the interior of $\xi\gamma$,

$$\xi\gamma\beta\delta_1 K \eta\gamma\beta\delta_1 K \xi\eta\beta\delta_1. \tag{4}$$

From (1) and (2), $\alpha\beta\gamma\delta_1 K \alpha\eta\gamma\delta_1$; from (1), (2), (3), $\alpha\beta\gamma\delta_1 K \alpha\beta\eta\delta_1$; from (1), (2), (3), (4), $\alpha\beta\gamma\delta_1 K \eta\beta\gamma\delta_1$. Hence η is in the interior of $\alpha\beta\gamma$.

Theorem 18. If $\alpha\beta\gamma\delta K$, then there is a ξ in the interior of $\alpha\beta\gamma$.

Proof follows at once from theorems 15 and 17.

Theorem 19. If $\alpha\beta\gamma\delta K \alpha'\gamma\beta\delta$, $\alpha\beta\alpha'\Delta\overline{K}$ for any Δ, and ξ is in the interior of $\beta\gamma\delta$, then ξ is in the interior of $\alpha\alpha'\gamma\delta$.

Since ξ is in the interior of $\beta\gamma\delta$ and $\alpha\beta\gamma\delta K$, $\alpha'\beta\gamma\delta K$, by definition 6, we have,

$$\alpha\beta\gamma\delta K \alpha\xi\gamma\delta K \alpha\beta\xi\delta K \alpha\beta\gamma\xi,$$
$$\alpha'\beta\gamma\delta K \alpha'\xi\gamma\delta K \alpha'\beta\xi\delta K \alpha'\beta\gamma\xi.$$

Since $\alpha\beta\gamma\delta K \alpha'\gamma\beta\delta$, we have $\alpha\beta\xi\delta K \alpha'\xi\beta\delta$, $\alpha\beta\gamma\xi K \alpha'\gamma\beta\xi$, $\alpha\xi\gamma\delta K \alpha'\gamma\xi\delta$. Also, since $\alpha\beta\gamma\delta K$, by theorem 13, $\alpha\beta\gamma\delta K \alpha'\beta\gamma\delta$ or $\alpha\beta\gamma\delta K \alpha\alpha'\gamma\delta$ or $\alpha\beta\gamma\delta K \alpha\beta\alpha'\delta$ or $\alpha\beta\gamma\delta K \alpha\beta\gamma\alpha'$.

Now $\alpha\beta\gamma\delta K \alpha'\gamma\beta\delta$, and hence $\alpha\beta\gamma\delta\overline{K}\alpha'\beta\gamma\delta$; and since $\alpha\beta\alpha'\Delta\overline{K}$ for any Δ, $\alpha\beta\gamma\delta\overline{K}\alpha\beta\alpha'\delta$ and $\alpha\beta\gamma\delta\overline{K}\alpha\beta\gamma\alpha'$. Hence we have $\alpha\beta\gamma\delta K \alpha\alpha'\gamma\delta$. In a similar manner it is shown that $\alpha\beta\xi\delta K$ and α' imply $\alpha\beta\xi\delta K \alpha\alpha'\xi\delta$, since $\alpha\beta\xi\delta K \alpha'\xi\beta\delta$ and $\alpha\beta\alpha'\Delta\overline{K}$ for any Δ; and that $\alpha\beta\gamma\xi K$ and α' imply $\alpha\beta\gamma\xi K \alpha\alpha'\gamma\xi$, since $\alpha\beta\gamma\xi K \alpha'\gamma\beta\xi$ and $\alpha\beta\alpha'\Delta\overline{K}$ for any Δ. Therefore, we have $\alpha\beta\gamma\delta K \alpha\alpha'\gamma\delta$, $\alpha\beta\xi\delta K \alpha\alpha'\xi\delta$, $\alpha\beta\gamma\xi K \alpha\alpha'\gamma\xi$. But from the preceding, $\alpha\beta\gamma\delta K \alpha\beta\xi\delta K \alpha\beta\gamma\xi K \alpha\xi\gamma\delta K \alpha'\gamma\xi\delta$. Hence

$$\alpha\alpha'\gamma\delta K \alpha\alpha'\xi\delta K \alpha\alpha'\gamma\xi K \alpha\xi\gamma\delta K \xi\alpha'\gamma\delta;$$

that is, ξ is in the interior of $\alpha\alpha'\gamma\delta$.

Theorem 20. If ξ is in the interior of $\beta\gamma\delta$ and η is in the interior of $\alpha\xi$ and $\alpha\beta\gamma\delta K$, then η is in the interior of $\alpha\beta\gamma\delta$.

Since $\alpha\beta\gamma\delta K$ and ξ is in the interior of $\beta\gamma\delta$, $\alpha\beta\gamma\delta K \alpha\xi\gamma\delta K \alpha\beta\xi\delta K \alpha\beta\gamma\xi$. Since η is in the interior of $\alpha\xi$ and $\alpha\xi\gamma\delta K$, $\alpha\beta\xi\delta K$, $\alpha\beta\gamma\xi K$, we have $\alpha\xi\gamma\delta K \alpha\eta\gamma\delta$, $\alpha\beta\xi\delta K \alpha\beta\eta\delta$, $\alpha\beta\gamma\xi K \alpha\beta\gamma\eta$. Hence

$$\alpha\eta\gamma\delta K \alpha\beta\eta\delta K \alpha\beta\gamma\eta K \alpha\beta\gamma\delta.$$

Now we have $\eta\beta\gamma\delta K$; for if $\eta\beta\gamma\delta\overline{K}$, then $\alpha\beta\gamma\delta K$, $\eta\beta\gamma\delta\overline{K}$, $\xi\beta\gamma\delta\overline{K}$ imply $\xi\eta\gamma\delta\overline{K}$, by theorem 7; but this contradicts $\xi\gamma\delta\alpha K\xi\gamma\delta\eta$, since η is in the interior of $\alpha\xi$. Since $\eta\beta\gamma\delta K$, we have, by axiom 16, $\alpha\beta\gamma\delta K\eta\beta\gamma\delta$ or $\alpha\beta\gamma\delta K\eta\gamma\beta\delta$.

We suppose $\alpha\beta\gamma\delta K\eta\gamma\beta\delta$. Then since $\eta\beta\gamma\delta K$ and ξ is in the interior of $\beta\gamma\delta$, we have $\eta\beta\gamma\delta K\eta\xi\gamma\delta K\eta\beta\xi\delta K\eta\beta\gamma\xi$; that is, $\eta\beta\gamma\delta K\eta\xi\gamma\delta$. But from the preceding, $\xi\gamma\delta\alpha K\beta\gamma\delta\alpha$, and therefore, since $\eta\gamma\beta\delta K\alpha\beta\gamma\delta$, $\eta\gamma\xi\delta K\alpha\xi\gamma\delta$. Thus $\xi\eta\alpha\Delta\overline{K}$ for any Δ, $\eta\gamma\xi\delta K\alpha\xi\gamma\delta$; also, by theorem 18, there exists a ζ in the interior of $\gamma\xi\delta$; thus we may apply theorem 19. By the latter theorem, ζ is in the interior of $\alpha\eta\gamma\delta$; hence, by axiom 15, there is a ξ' in the interior of $\alpha\eta$ and such that $\xi'\zeta\gamma\delta\overline{K}$. Let us suppose $\xi \neq \xi'$. Then since $\xi'\eta\alpha\Delta K_1$ and $\xi\eta\alpha\Delta K_1$ and $\xi \neq \xi'$, by theorem 11, we have $\alpha\xi\xi'\Delta K_1$. Since $\alpha\xi\gamma\delta K$, by axiom 7, $\xi'\xi\gamma\delta K$ or $\alpha\xi'\gamma\delta K$ or $\alpha\xi\xi'\delta K$ or $\alpha\xi\gamma\xi' K$; that is, since $\xi\xi'\gamma\delta\overline{K}$,* $\alpha\xi\xi'\delta\overline{K}$, $\alpha\xi\xi'\gamma\overline{K}$, $\alpha\xi'\gamma\delta K$. Then $\alpha\xi'\gamma\delta K$ and $\alpha\xi\gamma\delta K$ imply, by axiom 8, $\xi\xi'\alpha\gamma K$ or $\xi\xi'\alpha\delta K$ or $\xi\xi'\gamma\delta K$, which is impossible. Hence $\xi = \xi'$. That is, ξ is in the interior of $\alpha\eta$; but by hypothesis η is in the interior of $\alpha\xi$. Therefore, by theorem 15′, these statements are contradictory, and hence $\alpha\beta\gamma\delta\overline{K}\eta\gamma\beta\delta$; that is, $\alpha\beta\gamma\delta K\eta\beta\gamma\delta$. Hence we have $\alpha\eta\gamma\delta K\alpha\beta\eta\delta K\alpha\beta\gamma\eta K\alpha\beta\gamma\delta K\eta\beta\gamma\delta$; *i. e.*, η is in the interior of $\alpha\beta\gamma\delta$.

Theorem 21. If $\alpha\beta\gamma\delta K$ and ξ is in the interior of $\alpha\beta\gamma$, then there exists an η in the interior of $\beta\gamma$ such that $\alpha\delta\xi\eta K_2$.

By theorem 15 there exists a ζ in the interior of $\delta\xi$; therefore, by theorem 20, ζ is in the interior of $\alpha\beta\gamma\delta$. Hence, by axiom 15, there exists an η in the interior of $\beta\gamma$ such that $\alpha\delta\zeta\eta K_2$; that is, $\alpha\delta\xi\eta K_2$, by theorem 10.

Theorem 22. If $\alpha\beta\gamma\delta K\alpha'\gamma\beta\delta$ and $\alpha\alpha'\beta\Delta K_1$, then β is in the interior of $\alpha\alpha'$.

By theorem 18, there is a ξ in the interior of $\beta\gamma\delta$. Hence, by theorem 19, the point ξ is in the interior of $\alpha\alpha'\gamma\delta$. Hence, by axiom 15, there is a point β' in the interior of $\alpha\alpha'$ such that $\beta'\gamma\delta\xi\overline{K}$. Then, as in the proof of theorem 20 (in the case $\xi \neq \xi'$), we show that $\beta = \beta'$.

Theorem 23. If $\alpha\beta\gamma\delta K\alpha'\beta\delta\gamma$, then there is a ξ in the interior of $\alpha\alpha'$ such that $\beta\gamma\delta\xi K_2$.

Suppose, first, that $\alpha'\beta\gamma\delta K$, $\alpha\alpha'\gamma\delta K$, $\alpha\beta\alpha'\delta K$, $\alpha\beta\gamma\alpha' K$. We distinguish the following cases:

* Since $\xi\gamma\delta\alpha K$, $\zeta\xi\gamma\delta\overline{K}$, $\zeta\xi'\gamma\delta\overline{K}$, by theorem 7, $\xi\xi'\gamma\delta\overline{K}$.

CASE I. α' is in the interior of a compartment at a vertex of $\alpha\beta\gamma\delta$, say β.

Then $\alpha\beta\gamma\delta K\alpha'\beta\delta\gamma$, $\alpha\beta\gamma\delta K\alpha\alpha'\gamma\delta$, $\alpha\beta\gamma\delta K\alpha\beta\delta\alpha'$, $\alpha\beta\gamma\delta K\alpha\beta\alpha'\gamma$. That is, since $\alpha\alpha'\gamma\delta K$ and $\alpha\alpha'\gamma\delta K\beta\alpha'\gamma\delta$, $\alpha\alpha'\gamma\delta K\alpha\beta\gamma\delta$, $\alpha\alpha'\gamma\delta K\alpha\alpha'\beta\delta$, $\alpha\alpha'\gamma\delta K\alpha\alpha'\gamma\beta$, β is in the interior of $\alpha\alpha'\gamma\delta$. Hence, by axiom 15, there exists a ξ in the interior of $\alpha\alpha'$ and such that $\beta\gamma\delta\xi K_2$.

CASE II. α' is in the interior of a compartment at an edge of $\alpha\beta\gamma\delta$, say $\beta\gamma$.

Then $\alpha\beta\gamma\delta K\alpha'\beta\delta\gamma$, $\alpha\beta\gamma\delta K\alpha\alpha'\gamma\delta$, $\alpha\beta\gamma\delta K\alpha\beta\alpha'\delta$, $\alpha\beta\gamma\delta K\alpha\beta\alpha'\gamma$. Since $\beta\neq\gamma$, there exists, by theorem 15, a point ϕ in the interior of $\beta\gamma$, and we have, therefore, since $\alpha'\delta\beta\gamma K$, $\alpha'\alpha\beta\gamma K$, $\alpha\delta\beta\gamma K$,

$$\alpha'\delta\beta\gamma K\alpha'\delta\varphi\gamma,\quad \alpha'\alpha\beta\gamma K\alpha'\alpha\varphi\gamma,\quad \alpha\delta\beta\gamma K\alpha\delta\varphi\gamma,$$
$$\alpha'\delta\beta\gamma K\alpha'\delta\beta\varphi,\quad \alpha'\alpha\beta\gamma K\alpha'\alpha\beta\varphi,\quad \alpha\delta\beta\gamma K\alpha\delta\beta\varphi.$$

1) If $\alpha\alpha'\delta\phi K$, then we have $\alpha\beta\gamma\delta K\alpha\alpha'\delta\phi$ or $\alpha\beta\gamma\delta K\alpha'\alpha\delta\phi$. If $\alpha\beta\gamma\delta K\alpha\alpha'\delta\phi$, then ϕ is in the interior of $\alpha\alpha'\delta\beta$. For we have $\alpha\alpha'\delta\beta K$, and $\alpha'\beta\delta\gamma K\phi\alpha'\delta\beta$, $\alpha\beta\gamma\delta K\alpha\phi\delta\beta$, $\alpha\beta\alpha'\gamma K\alpha\alpha'\phi\beta$, $\alpha\beta\gamma\delta K\alpha\alpha'\delta\phi$. That is, since $\alpha\beta\gamma\delta K\alpha\beta\alpha'\gamma$ and $\alpha\beta\gamma\delta K\alpha'\beta\delta\gamma$, $\alpha\beta\gamma\delta K\alpha\alpha'\delta\beta$, we have $\alpha\alpha'\delta\beta K\phi\alpha'\delta\beta$, $\alpha\alpha'\delta\beta K\alpha\phi\delta\beta$, $\alpha\alpha'\delta\beta K\alpha\alpha'\phi\beta$, $\alpha\alpha'\delta\beta K\alpha\alpha'\delta\phi$. Thus ϕ is in the interior of $\alpha\alpha'\delta\beta$, and therefore, by axiom 15, there is a point η in the interior of $\alpha\alpha'$ such that $\beta\delta\phi\eta\overline{K}$; *i. e.*, $\beta\delta\phi\eta K_2$. Since $\beta\delta\phi\gamma K_2$, we have then, by theorem 7, $\beta\delta\gamma\eta\overline{K}$; *i. e.*, $\beta\gamma\delta\eta K_2$.

2) Let $\alpha\alpha'\delta\phi\overline{K}$. Then there is a point θ in the interior of $\phi\beta$. Therefore, θ is in the interior of $\beta\gamma$, by theorem 16. To show that $\alpha\beta\gamma\delta K\alpha\alpha'\delta\theta$. We have $\alpha'\delta\phi\beta K$, $\alpha\alpha'\phi\beta K$, $\alpha\delta\phi\beta K$. Hence

$$\alpha'\delta\varphi\beta K\alpha'\delta\theta\beta,\quad \alpha'\alpha\varphi\beta K\alpha'\alpha\theta\beta,\quad \alpha\delta\varphi\beta K\alpha\delta\theta\beta,$$
$$\alpha'\delta\varphi\beta K\alpha'\delta\varphi\theta,\quad \alpha'\alpha\varphi\beta K\alpha'\alpha\varphi\theta,\quad \alpha\delta\varphi\beta K\alpha\delta\varphi\theta.$$

Since $\alpha'\delta\phi\theta K$, we have, by theorem 13, $\alpha'\delta\phi\theta K\alpha\delta\phi\theta$ or $\alpha'\delta\phi\theta K\alpha\alpha'\phi\theta$ or $\alpha'\delta\phi\theta K\alpha'\delta\alpha\theta$ or $\alpha'\delta\phi\theta K\alpha'\delta\phi\alpha$.

Now we have $\alpha\delta\phi\theta K\alpha\delta\phi\beta$, $\alpha'\delta\phi\theta K\alpha'\delta\phi\beta$ and $\delta\alpha\phi\beta K\alpha'\delta\phi\beta$; hence $\alpha\delta\phi\theta K\alpha'\delta\theta\phi$. Also $\alpha'\delta\phi\theta K\alpha'\delta\phi\beta$, $\alpha'\alpha\phi\theta K\alpha'\alpha\phi\beta$ and $\alpha'\alpha\phi\beta K\delta\alpha'\phi\beta$; hence $\alpha'\delta\phi\theta K\alpha'\alpha\theta\phi$. Finally, we have $\alpha'\alpha\delta\phi\overline{K}$. Hence, we have $\alpha'\delta\phi\theta K\alpha'\delta\alpha\theta$.

But $\alpha'\delta\phi\beta K\alpha'\delta\phi\theta$ and $\alpha\beta\gamma\delta K\phi\alpha'\delta\beta$. Hence $\alpha\beta\gamma\delta K\alpha'\delta\alpha\theta$; *i. e.*, $\alpha\beta\gamma\delta K\alpha\alpha'\delta\theta$, and we may proceed as under 1).

CASE III. α' is in the interior of the compartment at the face $\beta\gamma\delta$ of $\alpha\beta\gamma\delta$.

From the hypothesis, we have $\alpha\beta\gamma\delta K\alpha'\gamma\beta\delta$, and $\alpha\beta\gamma\delta K\alpha\alpha'\gamma\delta$, $\alpha\beta\gamma\delta K\alpha\beta\alpha'\delta$, $\alpha\beta\gamma\delta K\alpha\beta\gamma\alpha'$. Since $\beta \neq \gamma$, there exists a θ in the interior of $\beta\gamma$, by theorem 15. Then since $\alpha\beta\gamma\delta K$, $\alpha\beta\gamma\alpha' K$ and $\alpha'\beta\gamma\delta K$, we have

$$\alpha\beta\gamma\delta K\alpha\theta\gamma\delta, \quad \alpha'\gamma\beta\delta K\alpha'\theta\beta\delta, \quad \alpha\beta\gamma\alpha' K\alpha\theta\gamma\alpha',$$
$$\alpha\beta\gamma\delta K\alpha\beta\theta\delta, \quad \alpha'\gamma\beta\delta K\alpha'\gamma\theta\delta, \quad \alpha\beta\gamma\alpha' K\alpha\beta\theta\alpha'.$$

1) If $\alpha\alpha'\delta\theta K$, then $\alpha\beta\gamma\delta K\alpha\alpha'\delta\theta$ or $\alpha\beta\gamma\delta K\alpha'\alpha\delta\theta$. If $\alpha\beta\gamma\delta K\alpha\theta\alpha'\delta$ then α' is in the interior of the compartment at $\theta\delta$ of $\theta\beta\alpha\delta$. For we have $\alpha\beta\gamma\delta K\alpha\beta\theta\delta$, and hence $\alpha\beta\theta\delta K$. Also since $\alpha\beta\gamma\delta K\alpha'\gamma\beta\delta$ and $\alpha'\gamma\beta\delta K\alpha'\theta\beta\delta$, we have $\alpha\beta\gamma\delta K\alpha'\theta\beta\delta$. Also $\alpha\beta\gamma\delta K\alpha\theta\alpha'\delta$ and $\alpha\beta\gamma\delta K\alpha\beta\alpha'\delta$. Finally, since $\alpha\beta\gamma\delta K\alpha\beta\gamma\alpha'$ and $\alpha\beta\gamma\alpha' K\alpha\beta\theta\alpha'$, we have $\alpha\beta\gamma\delta K\alpha\beta\theta\alpha'$. Hence $\alpha\beta\theta\delta K\alpha'\theta\beta\delta$, $\alpha\beta\theta\delta K\alpha\theta\alpha'\delta$, $\alpha\beta\theta\delta K\alpha\beta\alpha'\delta$, $\alpha\beta\theta\delta K\alpha\beta\theta\alpha'$. Thus we may proceed further as under Case II, 1). A similar discussion is valid if $\alpha\beta\gamma\delta K\alpha\theta\delta\alpha'$.

2) Let $\alpha\alpha'\delta\theta\overline{K}$. Then, by theorem 15, there is a θ' in the interior of $\theta\delta$, then, by theorem 17, θ' is in the interior of $\beta\gamma\delta$. Hence, by theorem 21, there is a θ'' in the interior of $\beta\delta$ such that $\gamma\alpha\theta'\theta'' K_2$. If $\gamma\alpha\theta''\alpha' K$, then we proceed as under 1). Suppose $\gamma\alpha\theta''\alpha'\overline{K}$; then since $\alpha\delta\theta\alpha'\overline{K}$, $\alpha\delta\theta\theta'\overline{K}$ and $\alpha\delta\theta\beta K$, we have, by theorem 7, $\alpha'\theta'\alpha\delta\overline{K}$. Similarly, $\gamma\alpha\theta''\alpha'\overline{K}$, $\gamma\alpha\theta''\theta'\overline{K}$, $\gamma\alpha\theta''\beta K$ imply $\alpha'\theta'\alpha\theta''\overline{K}$. Since $\theta'\theta''\delta\alpha K$ and $\alpha'\theta'\alpha\delta\overline{K}$, $\alpha'\theta'\alpha\theta''\overline{K}$, $\alpha\alpha'\theta'\Delta\overline{K}$ for any Δ. Moreover, since θ' is in the interior of $\beta\gamma\delta$, we have $\beta\theta'\delta\alpha K\theta'\beta\delta\alpha'$; and, by theorem 18, there is a ξ in the interior of $\beta\theta'\delta$. Hence, by theorem 19, ξ is in the interior of $\alpha\alpha'\beta\delta$.*

Suppose, secondly, that $\alpha'\beta\gamma\delta\overline{K}$ or $\alpha\alpha'\gamma\delta\overline{K}$ or $\alpha\beta\alpha'\delta\overline{K}$ or $\alpha\beta\gamma\alpha'\overline{K}$. We distinguish two cases:

CASE I. $\alpha'\beta\gamma\delta K$, $\alpha\alpha'\gamma\delta K$, $\alpha\beta\alpha'\delta K$, $\alpha\beta\gamma\alpha'\overline{K}$.

Since $\beta \neq \delta$, there exists a ϕ in the interior of $\beta\delta$. Then since $\alpha'\gamma\beta\delta K$, $\alpha\alpha'\beta\delta K$, $\alpha\gamma\beta\delta K$, we have

$$\alpha'\gamma\beta\delta K\alpha'\gamma\varphi\delta, \quad \alpha\alpha'\beta\delta K\alpha\alpha'\varphi\delta, \quad \alpha\gamma\beta\delta K\alpha\gamma\varphi\delta,$$
$$\alpha'\gamma\beta\delta K\alpha'\gamma\beta\varphi, \quad \alpha\alpha'\beta\delta K\alpha\alpha'\beta\varphi, \quad \alpha\gamma\beta\delta K\alpha\gamma\beta\varphi.$$

Hence $\alpha'\gamma\delta\phi K\alpha\delta\gamma\phi$. Also $\alpha'\delta\gamma\phi K$, $\alpha\delta\alpha'\phi K$, $\alpha\delta\gamma\alpha' K$. Further $\alpha\alpha'\gamma\phi K$. For if $\alpha\alpha'\gamma\phi\overline{K}$, since $\alpha\beta\gamma\alpha'\overline{K}$ and $\alpha\gamma\beta\phi K$, then $\alpha\gamma\alpha'\Delta\overline{K}$ for any Δ, by theorem 6. Hence $\alpha\gamma\alpha'\delta\overline{K}$, which contradicts the hypothesis. Hence $\alpha\alpha'\gamma\phi K$, and this case is reduced to the previous discussion.

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* Since $\alpha\alpha'\theta'\Delta\overline{K}_1$ and $\beta\theta'\delta\alpha K\theta'\beta\delta\alpha'$, by theorem 22, θ' is in the interior of $\alpha\alpha'$; since $\theta'\beta\gamma\delta\overline{K}$, θ' is the point required by theorem.

CASE II. $\alpha' \beta \gamma \delta K$, $\alpha \alpha' \gamma \delta K$, $\alpha \beta \alpha' \delta \overline{K}$, $\alpha \beta \gamma \alpha' \overline{K}$.

Since $\alpha \beta \gamma \delta K$, $\alpha \beta \alpha' \delta \overline{K}$, $\alpha \beta \gamma \alpha' \overline{K}$, then $\alpha \beta \alpha' \Delta \overline{K}$ for any Δ. By theorem 18, there is a ξ in the interior of $\beta \gamma \delta$. Also, by hypothesis, $\alpha \beta \gamma \delta K \alpha' \gamma \beta \delta$. Then we can apply theorem 19 to show that ξ is in the interior of $\alpha \alpha' \gamma \delta$.

Theorem 24. If $\xi \beta \gamma \delta K \alpha \xi \gamma \delta$, $\xi \beta \gamma \delta K \alpha \beta \xi \delta$ and $\alpha \beta \gamma \xi \overline{K}$, then ξ is in the interior of $\alpha \beta \gamma$.

By hypothesis $\gamma \xi \beta \delta K \alpha \beta \xi \delta$, and hence, by theorem 23, there is an η in the interior of $\alpha \gamma$ such that $\xi \beta \delta \eta \overline{K}$. Therefore, since $\alpha \gamma \xi \delta K$ and $\alpha \beta \gamma \delta K$, we have

$$\eta \gamma \xi \delta K \alpha \eta \xi \delta K \alpha \gamma \xi \delta, \tag{1}$$

$$\eta \gamma \beta \delta K \alpha \eta \beta \delta K \alpha \gamma \beta \delta. \tag{2}$$

From the hypothesis and axiom 10′, we have $\alpha \xi \gamma \delta K \alpha \beta \xi \delta$, and from (1) we have $\alpha \xi \eta \delta K \alpha \xi \gamma \delta$; hence $\eta \alpha \xi \delta K \beta \xi \alpha \delta$. Further, we have from (2), $\alpha \beta \eta \delta K$; also $\beta \eta \xi \delta \overline{K}$, and since $\alpha \beta \gamma \xi K_2$ and $\alpha \gamma \eta \Delta K_1$, we have, by theorem 9, $\alpha \beta \xi \eta \overline{K}$. Hence, by theorem 6, $\beta \xi \eta \Delta \overline{K}$ for any Δ. Since $\eta \alpha \xi \delta K \beta \xi \alpha \delta$ and $\beta \xi \eta \Delta K$, by theorem 22, ξ is in the interior of $\beta \eta$. Since $\alpha \beta \gamma \delta K$ and η is in the interior of $\alpha \gamma$ and ξ is in the interior of $\beta \eta$, ξ is in the interior of $\alpha \beta \gamma$ by theorem 17.

Theorem 25. If $\alpha \beta \gamma \delta K$, then there exists a ξ such that $\alpha \beta \gamma \delta K \beta \alpha \xi \delta$ and $\alpha \gamma \xi \Delta K_1$.

Since $\alpha \beta \gamma \delta K$, by axiom 14, there is a point η such that $\alpha \beta \gamma \delta K \eta \alpha \gamma \delta$, $\alpha \beta \gamma \delta K \eta \beta \alpha \delta$, $\alpha \beta \gamma \delta K \eta \beta \gamma \alpha$. That is, by axiom 10, $\alpha \beta \gamma \delta K \eta \beta \gamma \delta$. Since $\alpha \beta \gamma \delta K \eta \beta \gamma \alpha$, by theorem 23, there is an η' in the interior of $\eta \delta$ such that $\alpha \beta \gamma \eta' \overline{K}$.* Since $\eta \delta \beta \alpha K$, $\eta \delta \gamma \beta K$, $\eta \delta \alpha \gamma K$, we have then $\eta \delta \beta \alpha K \eta' \delta \beta \alpha$, $\eta \delta \gamma \beta K \eta' \delta \gamma \beta$, $\eta \delta \alpha \gamma K \eta' \delta \alpha \gamma$. Since $\alpha \beta \gamma \delta K \eta \alpha \gamma \delta$ and $\eta \delta \alpha \gamma K \eta' \delta \alpha \gamma$, we have $\alpha \beta \gamma \delta K \eta' \delta \alpha \gamma$; that is, $\beta \gamma \alpha \delta K \eta' \alpha \gamma \delta$. Hence, by theorem 23, there is a ξ in the interior of $\eta' \beta$ such that $\alpha \gamma \delta \xi \overline{K}$. Now since $\eta' \beta \delta \alpha K$, we have $\eta' \beta \delta \alpha K \xi \beta \delta \alpha$; also $\eta \delta \beta \alpha K \eta' \delta \beta \alpha$ and $\alpha \beta \gamma \delta K \eta \beta \alpha \delta$. Hence $\beta \alpha \gamma \delta K \xi \beta \delta \alpha$; *i. e.*, $\alpha \beta \gamma \delta K \beta \alpha \xi \delta$. Moreover, we have $\eta' \beta \xi \Delta K_1$ and $\alpha \beta \gamma \eta' K_2$. Hence $\eta' \beta \xi \alpha \overline{K}$, $\eta' \beta \alpha \xi K_2$ and $\eta' \beta \alpha \gamma K_2$; therefore, by theorem 7, $\alpha \beta \gamma \xi \overline{K}$; *i. e.*, $\alpha \beta \gamma \xi K_2$. Since $\alpha \beta \gamma \xi K_2$, $\alpha \gamma \delta \xi K_2$ and $\alpha \beta \gamma \delta K$, we have, by theorem 6, $\alpha \gamma \xi \Delta K_1$.

Theorem 26. If α, β are distinct, then there is a ξ such that β is in the interior of $\alpha \xi$.

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* This statement follows also by the direct application of axiom 15, since α is in the interior of $\eta \beta \gamma \delta$.

By theorem 5 and axiom 2, there are two points γ, δ such that $\alpha\beta\gamma\delta K$. By theorem 25, there is a ξ such that $\alpha\beta\xi\Delta K_1$ and $\alpha\beta\gamma\delta K\xi\gamma\beta\delta$. Hence. by theorem 22, β is in the interior of $\alpha\xi$.

Theorem 27. If $\alpha\beta\xi\Delta K_1$ and α, β, ξ are distinct, then one of the points α, β, ξ is in the interior of the other two points.

Since $\alpha \neq \beta$, there exist two points γ, δ such that $\alpha\beta\gamma\delta K$. Then since $\alpha\beta\xi\Delta K_1$ and $\xi \neq \alpha, \beta$, we have $\gamma\alpha\xi\delta K$, $\gamma\beta\xi\delta K$, by theorem 8. Then, by axiom 16, we consider these cases:

$$\beta\gamma\xi\delta K\beta\gamma\alpha\delta, \quad \gamma\alpha\xi\delta K\gamma\alpha\beta\delta. \tag{1}$$

By theorem 22, ξ is in the interior of $\alpha\beta$.

$$\beta\gamma\xi\delta K\beta\gamma\alpha\delta, \quad \gamma\alpha\xi\delta K\alpha\gamma\beta\delta. \tag{2}$$

Then α is in the interior of $\xi\beta$.

$$\beta\gamma\xi\delta K\gamma\beta\alpha\delta, \quad \gamma\alpha\xi\delta K\gamma\alpha\beta\delta. \tag{3}$$

Then β is in the interior of $\xi\alpha$.

$$\beta\gamma\xi\delta K\gamma\beta\alpha\delta, \quad \gamma\alpha\xi\delta K\alpha\gamma\beta\delta. \tag{4}$$

Hence $\beta\gamma\xi\delta K\gamma\alpha\xi\delta$; that is, $\beta\delta\gamma\xi K\alpha\gamma\delta\xi$. Therefore, by theorem 22, ξ is in the interior of $\alpha\beta$. That is, since $\alpha\beta\gamma\delta K$, $\alpha\beta\gamma\delta K\xi\beta\gamma\delta K\alpha\xi\gamma\delta$. Since $\gamma\alpha\xi\delta K\alpha\beta\gamma\delta$ and $\gamma\alpha\xi\delta K\beta\alpha\gamma\delta$, we have $\alpha\beta\gamma\delta K\beta\alpha\gamma\delta$, which contradicts axiom 4. Thus case (4) is impossible.

Theorem 28. If $\alpha\beta\gamma\delta K\alpha\beta\gamma\delta'$, then there is no point ξ in the interior of $\delta\delta'$ such that $\alpha\beta\gamma\xi\bar{K}$.

Let a point ξ be in the interior of $\delta\delta'$, supposing that $\delta \neq \delta'$. Then since $\alpha\beta\gamma\delta K$, $\alpha\beta\gamma\delta' K$ and $\delta \neq \delta'$, we have, by axiom 8, $\delta'\beta\gamma\delta K$ or $\alpha\delta'\gamma\delta K$ or $\alpha\beta\delta'\delta K$. Let $\alpha\delta'\gamma\delta K$. By theorem 25, there exists a δ'' such that $\delta\gamma\delta''\Delta K_1$ and $\alpha\beta\gamma\delta K\beta\alpha\gamma\delta''$, and by theorem 22, γ is in the interior of $\delta\delta''$. Since $\alpha\beta\gamma\delta K\beta\alpha\gamma\delta''$ and $\alpha\beta\gamma\delta K\alpha\beta\gamma\delta'$, $\alpha\beta\gamma\delta' K\beta\alpha\gamma\delta''$ and hence there is a point η in the interior of $\delta'\delta''$ such that $\alpha\beta\gamma\eta\bar{K}$, by theorem 23. Now $\delta\delta'\delta''\alpha K$, for if $\delta\delta'\delta''\alpha\bar{K}$, $\delta\delta'\delta''\gamma\bar{K}$, $\alpha\delta'\gamma\delta K$ imply $\delta\delta'\delta''\Delta K_1$; also $\delta\delta''\gamma\Delta K_1$, and hence, by theorem 11, $\delta\delta'\gamma\Delta K_1$, which contradicts $\delta\delta'\gamma\alpha K$,

To prove the above theorem, we show then that if $\delta\delta'\delta''\alpha K$, γ is in the interior of $\delta\delta''$, η is in the interior of $\delta'\delta''$, and ξ is in the interior of $\delta\delta'$, then $\alpha\gamma\eta\xi K$. Suppose $\alpha\gamma\eta\xi\bar{K}$; then since $\alpha\delta'\delta''\delta K\alpha\delta'\gamma\delta K\alpha\delta'\delta''\gamma$, we have $\alpha\gamma\delta'\delta'' K$, and hence $\alpha\gamma\delta'\delta'' K\alpha\gamma\eta\delta'' K\alpha\gamma\delta'\eta$; *i. e.*, $\alpha\gamma\delta'\eta K$. Also, since $\delta\delta'\delta''\xi\bar{K}$ and γ is in the interior of $\delta\delta''$, we have $\delta'\delta''\gamma\xi\bar{K}$; and since

$\delta''\delta'\gamma\alpha K$, η is in the interior of $\delta'\delta''$ and $\delta'\delta''\gamma\xi\bar{K}$, we have $\delta'\gamma\eta\xi\bar{K}$. Thus $\alpha\gamma\delta'\eta K$, $\delta'\gamma\eta\xi\bar{K}$, $\alpha\gamma\eta\xi\bar{K}$; therefore, by theorem 6, $\gamma\eta\xi\Delta K_1$. Hence, by theorem 27, since γ, η, ξ are distinct, ξ is in the interior of $\eta\gamma$, or γ is in the interior of $\eta\xi$, or η is in the interior of $\xi\gamma$. Since $\delta\delta'\delta''\alpha K$ and the ξ, η, γ are symmetrically involved with reference to $\delta\delta'\delta''\alpha$, it will suffice to prove that ξ is not in the interior of $\eta\gamma$. Suppose the contrary. Since $\delta\delta'\delta''\alpha K$ and γ is in the interior of $\delta\delta''$, we have $\delta\delta'\delta''\alpha K\gamma\delta'\delta''\alpha K\delta\delta'\gamma\alpha$. Since $\delta'\gamma\delta''\alpha K$ and η is in the interior of $\delta'\delta''$ and ξ is in the interior of $\gamma\eta$, by theorem 17, ξ is in the interior of $\gamma\delta'\delta''$; therefore, $\gamma\delta'\delta''\alpha K\xi\delta'\delta''\alpha K\gamma\xi\delta''\alpha K\gamma\delta'\xi\alpha$. But since ξ is in the interior of $\delta\delta'$ and $\delta\delta'\gamma\alpha K$, we have $\delta\delta'\gamma\alpha K\xi\delta'\gamma\alpha K\delta\xi\gamma\alpha$. Hence $\delta\delta'\gamma\alpha K\xi\delta'\gamma\alpha$ and $\gamma\delta'\delta''\alpha K\gamma\delta'\xi\alpha$. Therefore $\gamma\delta'\delta''\alpha\bar{K}\delta\delta'\gamma\alpha$. The remaining cases are proved in a similar manner. Therefore theorem 27 is contradicted. Hence $\alpha\gamma\xi\eta K$.

Since $\alpha\beta\gamma\delta K$, $\alpha\beta\gamma\eta\bar{K}$, $\alpha\gamma\xi\eta K$, then $\alpha\beta\gamma\xi K$. For if $\alpha\beta\gamma\xi\bar{K}$, since $\alpha\gamma\xi\eta K$ and $\alpha\beta\gamma\eta\bar{K}$, then $\alpha\beta\gamma\Delta\bar{K}$ for any Δ, which contradicts $\alpha\beta\gamma\delta K$. Thus if ξ is in the interior of $\delta\delta'$ and $\alpha\beta\gamma\delta K\alpha\beta\gamma\delta'$, then $\alpha\beta\gamma\xi K$, which establishes our theorem.

Theorem 29. If $\alpha\beta\gamma\xi K_2$ and $\delta\xi\delta'\xi K_2$, there exists an $\eta \neq \xi$ such that $\alpha\beta\gamma\eta K_2$ and $\delta\xi\delta'\eta K_2$.

If $\alpha\beta\gamma\delta K_2$, then since $\delta\xi\delta'\delta K_2$ and $\delta \neq \xi$, δ is the point required. If $\alpha\beta\gamma\delta' K_2$, then since $\delta\xi\delta'\delta K_2$ and $\delta' \neq \xi$, δ' is the point required. Let $\alpha\beta\gamma\delta K$ and $\alpha\beta\gamma\delta' K$. Then we distinguish the following cases:

CASE I. $\alpha\beta\gamma\delta K\beta\alpha\gamma\delta'$.

Then by theorem 23 there is a point η in the interior of $\delta\delta'$ such that $\alpha\beta\gamma\eta K_2$. Since $\delta'\delta\eta\Delta K_1$ and $\delta\xi\delta'\xi K_2$, we have $\delta\delta'\eta\xi\bar{K}$ and $\delta\xi\delta'\eta K_2$. Also $\xi \neq \eta$; for if $\xi = \eta$, then $\delta\delta'\xi\Delta K_1$, which contradicts $\delta\delta'\xi\xi K_2$.

CASE II. $\alpha\beta\gamma\delta K\alpha\beta\gamma\delta'$.

Then there is a δ'' such that $\xi\delta\delta''\Delta K_1$ and $\alpha\beta\xi\delta K\beta\alpha\xi\delta''$, by theorem 25, if we suppose $\alpha\beta\xi\delta K$. The latter assumption is permissible, since $\alpha\beta\gamma\delta K$, $\xi \neq \alpha$ or β; if $\xi \neq \alpha$, then, by theorem 5, $\xi\alpha\beta\gamma K$ or $\xi\alpha\beta\delta K$ or $\xi\alpha\gamma\delta K$. Since $\xi\alpha\beta\gamma\bar{K}$, $\xi\alpha\beta\delta K$ or $\xi\alpha\gamma\delta$; we suppose, as indicated above, $\xi\alpha\beta\delta K$. Now $\alpha\beta\xi\delta' K$; for if $\alpha\beta\xi\delta'\bar{K}$, since $\alpha\beta\xi\delta' K_2$ and $\alpha\beta\xi\gamma K_2$, we have, by theorem 7, $\alpha\beta\gamma\delta'\bar{K}$. Also $\alpha\beta\xi\delta' K\alpha\beta\xi\delta$. For otherwise, $\alpha\beta\xi\delta' K\beta\alpha\xi\delta$. Hence, by theorem 23, there is a point η' in the interior of $\delta\delta'$ such that $\alpha\beta\xi\eta' K_2$; that is, since $\alpha\beta\xi\gamma K_2$, by theorem 7, $\alpha\beta\gamma\eta' K_2$. But since $\alpha\beta\gamma\delta K\alpha\beta\gamma\delta'$, there is no η' in the interior of $\delta\delta'$ such that $\alpha\beta\gamma\eta' K_2$, by

theorem 28. Hence $\alpha\beta\xi\delta' K \alpha\beta\xi\delta$. Since, further, $\alpha\beta\xi\delta K \beta\alpha\xi\delta''$, we have $\alpha\beta\xi\delta' K \beta\alpha\xi\delta''$. Therefore, there is a point η in the interior of $\delta'\delta''$ such that $\alpha\beta\xi\eta K_2$. Since $\alpha\beta\xi\eta K_2$ and $\alpha\beta\xi\gamma K_2$, we have, by theorem 7, $\alpha\beta\gamma\eta K_2$. Also $\delta\xi\delta'\xi K_2$ and $\delta\xi\delta''\Delta K_1$; hence $\delta\xi\delta''\delta'\overline{K}$; that is, $\delta\xi\delta'\delta'' K_2$. Since $\delta'\xi\delta\delta'' K_2$ and $\delta'\delta''\eta\Delta K_1$, we have, by theorem 9, $\delta'\xi\delta\eta K_2$. Also $\xi \neq \eta$, for otherwise $\delta\xi\delta''\Delta K_1$ and $\delta''\xi\delta'\Delta K_1$ imply, by theorem 11, $\delta\delta'\xi\Delta K_1$, which contradicts $\delta\xi\delta'\xi K_2$.

Theorem 30. If $\alpha\beta\gamma\delta K$, ξ is in the interior of $\alpha\beta$, $\alpha\beta\gamma\eta K_2$, $\xi\eta\alpha\alpha K_2$, $\xi\eta\beta\beta K_2$, $\xi\eta\gamma\gamma K_2$, then there is a ζ such that $\xi\eta\zeta\Delta K_1$, and ζ is in the interior of $\alpha\gamma$ or $\beta\gamma$.

To prove this theorem it will suffice to show that $\xi\eta\delta\alpha K \eta\xi\delta\gamma$ or $\xi\eta\delta\beta K \eta\xi\delta\gamma$. We distinguish two cases.

CASE I. $\alpha\gamma\eta\Delta K_1$ or $\beta\gamma\eta\Delta K_1$.

Then since α, γ, η are distinct, by theorem 27, if $\alpha\gamma\eta\Delta K_1$, γ is in the interior of $\alpha\eta$, or η is in the interior of $\alpha\gamma$, or α is in the interior of $\eta\gamma$. If η is in the interior of $\alpha\gamma$, then we take $\zeta = \eta$. Let γ be in the interior of $\alpha\eta$. Then since $\alpha\eta\beta\delta K$ and $\alpha\eta\xi\delta K$, by theorem 8, and γ is in the interior of $\alpha\eta$,

$$\alpha\eta\beta\delta K \gamma\eta\beta\delta K \alpha\gamma\beta\delta, \tag{1}$$
$$\alpha\eta\xi\delta K \gamma\eta\xi\delta K \alpha\gamma\xi\delta, \tag{2}$$

and since $\alpha\beta\eta\delta K$, $\alpha\beta\gamma\delta K$, and ξ is in the interior of $\alpha\beta$,

$$\alpha\beta\eta\delta K \xi\beta\eta\delta K \alpha\xi\eta\delta, \tag{3}$$
$$\alpha\beta\gamma\delta K \xi\beta\gamma\delta K \alpha\xi\gamma\delta. \tag{4}$$

From (1) and (4), we have $\alpha\eta\beta\delta K \alpha\gamma\xi\delta$; from (2), $\gamma\eta\xi\delta K \alpha\gamma\xi\delta$; from (3), $\alpha\beta\eta\delta K \xi\beta\eta\delta$. Hence $\alpha\eta\beta\delta K \alpha\gamma\xi\delta K \gamma\eta\xi\delta K \xi\eta\beta\delta$. That is, $\gamma\eta\xi\delta K \xi\eta\beta\delta$; *i. e.*, $\eta\xi\delta\gamma K \xi\eta\delta\beta$.

If α is in the interior of $\eta\gamma$, then since $\eta\gamma\beta\delta K$ and ξ is in the interior of $\alpha\beta$, by theorem 17, ξ is in the interior of $\gamma\beta\eta$. Therefore,

$$\gamma\beta\eta\delta K \xi\beta\eta\delta K \gamma\xi\eta\delta K \gamma\beta\xi\delta.$$

Hence $\xi\beta\eta\delta K \gamma\xi\eta\delta$; *i. e.*, $\xi\eta\delta\beta K \eta\xi\delta\gamma$.

CASE II. $\alpha\gamma\eta\eta K_2$ and $\beta\gamma\eta\eta K_2$.

Then, by referring to the proof of theorem 14, it can easily be shown that either (1) η is in the interior of $\alpha\beta\gamma$ or (2) $\alpha\beta\gamma\delta K \eta\alpha\gamma\delta K \alpha\beta\eta\delta$ or (3) $\alpha\beta\gamma\delta K \eta\beta\gamma\delta K \alpha\beta\eta\delta$ or (4) $\alpha\beta\gamma\delta K \beta\eta\gamma\delta K \alpha\eta\gamma\delta$.

If η is in the interior of $\alpha\beta\gamma$, then $\alpha\beta\gamma\delta K\eta\beta\gamma\delta K\alpha\eta\gamma\delta K\alpha\beta\eta\delta$. Since $\alpha\beta\gamma\delta K$ and $\alpha\beta\eta\delta K$ and ξ is in the interior of $\alpha\beta$, we have $\alpha\beta\gamma\delta K\xi\beta\gamma\delta K\alpha\xi\gamma\delta$ and $\alpha\beta\eta\delta K\xi\beta\eta\delta K\alpha\xi\eta\delta$. Since $\xi\beta\gamma\delta K$ and $\xi\gamma\eta\beta\bar{K}$, we have $\xi\gamma\delta\eta K$; for if $\xi\gamma\delta\eta\bar{K}$, then $\xi\gamma\eta\Delta K_1$, which contradicts $\xi\eta\gamma\gamma K_2$. Therefore $\xi\gamma\delta\eta K\alpha\xi\gamma\delta$ or $\xi\gamma\delta\eta K\alpha\gamma\xi\delta$. Suppose $\eta\xi\gamma\delta K\alpha\xi\gamma\delta$. Then we have $\alpha\beta\gamma\delta K\alpha\beta\eta\delta$, $\alpha\beta\eta\delta K\alpha\xi\eta\delta$, $\alpha\beta\gamma\delta K\alpha\xi\gamma\delta$, $\alpha\xi\gamma\delta K\eta\xi\gamma\delta$. Hence $\alpha\xi\eta\delta K\eta\xi\gamma\delta$; that is, $\xi\eta\delta\alpha K\eta\xi\delta\gamma$. Similarly if $\eta\xi\gamma\delta K\alpha\gamma\xi\delta$.

The remaining sub-cases (2)–(4) are proved in an analogous manner.

The preceding descriptive theorems show us at once that the relation K, as satisfying the system ${}^3K_3$, suffices to generate projective geometry of three dimensions if one adds to the system ${}^3K_3$ an axiom of continuity. For we may easily identify theorems which we have proved with Hilbert's axioms of connection and order.* Thus our definition of $\alpha\beta\gamma\delta K_1$ corresponds to Hilbert's I, 1; theorem 11 corresponds to I, 2; our definition of $\alpha\beta\gamma\delta K_2$ corresponds to I, 3; theorem 12 corresponds to I, 4; theorem 10 corresponds to I, 5; theorem 29 corresponds to I, 6; theorem 15′ corresponds to II, 1; theorems 15 and 26 correspond to II, 2; theorem 27 corresponds to II, 3; theorem 30 corresponds to II, 5. It is not necessary to prove a theorem corresponding to Hilbert's II, 4, since the latter axiom, as part of Hilbert's system, has been proved redundant.† A direct proof of II, 4 on the basis of our axioms can be very easily given.

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* Compare Hilbert's "Foundations of Geometry," translated by E. J. Townsend.

† E. H. Moore, *Transactions American Math. Soc.*, 1902, pp. 142–158, 501; R. L. Moore, *American Math. Monthly*, April, 1902.